On disjunction and numerical existence properties of extensions of Heyting arithmetic

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Vorgelegte Bachelor Thesis von Alexej Gossmann

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(Alexej Gossmann)

Abstract

This bachelor thesis explores the relation between the disjunction and the numerical existence properties of extensions of arithmetic. It is shown that a recursively enumerable extension of (intuitionistic) arithmetic fulfills the disjunction property if and only if it also fulfills the numerical existence property. An analog result is shown for a modified version of these properties too, namely with a premise inserted in front of the disjunction and the existence statements. Both proofs, which interestingly are based on certain self referential sentences, are worked out in great detail. An analysis of the first proof, which is based on Friedman's proof given in [Fri75], discusses among other topics the establishment of the numerical existence property for classes Σ^0_n and Π^0_n of formulae, as well as for Burr's classification into Φ_n-formulae ([Bur00]), which is more appropriate for intuitionistic logic. In this context, based on [Lei85], it is also shown, that HA proves, that the disjunction property for disjunctions of Σ^0_2-sentences implies the numerical existence property of HA. Furthermore it is shown that if a recursively enumerable extension of arithmetic proves its own disjunction property, then it also proves its own inconsistency. This theorem, which again is due to [Fri75], is proven in great detail as well.

Contents

1 Introduction

This thesis is essentially based on the article "The disjunction property implies the numerical existence property" (denoted here by [Fri75]) authored by Harvey Friedman. This very short article was published in *Proceedings of the National Academy of Sciences of the United States of America* in 1975 after being communicated by Kurt Gödel (which indeed is remarkable).
The article is concerned with the disjunction and the numerical existence properties of extensions of intuitionistic arithmetic. A theory T is said to have the disjunction property, if for all sentences A and B the statement $\mathrm{T} \vdash A \vee B$ implies that either $\mathrm{T} \vdash A$ or $\mathrm{T} \vdash B$ holds. T is said to have the numerical existence property, if for any formula $C(x)$, with no free variable other than x, the statement $\mathrm{T} \vdash \exists x C(x)$ implies that there is a number n, such that $T \vdash C(\overline{n})$ holds.
Friedman noticed that the disjunction and the numerical existence properties were established for various extensions of arithmetic, such as HA itself, theories of functionals, second order arithmetic, the theory of types, and set theories. Interestingly, in each case the numerical existence property was shown utilizing the same methods that were used to show the disjunction property, such as cut elimination, normalization, Kripke models, Kleene's realizabilities, and Kleene's |. This led to the conjecture that in an arbitrary recursively enumerable extension of (intuitionistic) arithmetic the disjunction property generally implies the numerical existence property (the converse is quite trivial). This conjecture as well as its proof is the main topic of the article [Fri75].
After a short introduction of required definitions and lemmata in section 2, section 3 presents the above-mentioned theorem with a detailed proof. Not only is the proof interesting because of the result it establishes, but also because of the technique which it uses. Interestingly, Friedman's proof is based on several self referential sentences. But unlike the well-known proofs of the incompleteness theorems, which too are based on self referential sentences, Friedman uses such sentences to show a "positive" statement. That is, to establish the numerical existence property, which then implies the provability of many sentences. In contrast, self referential sentences are usually employed to show something "negative", as for instance in the case of Gödel's second incompleteness theorem to show the unprovability of the sentence claiming the consistency of the theory.
Section 4 presents some corollaries of the proof given in section 3. Derivation of algorithms for the witness number from that proof is discussed; that is, algorithms to compute a witness number (or its upper bound) for an existence statement out of the Gödel number of the proof of this statement. Furthermore, we address the question of how the assumptions concerning the disjunction property can be weakened, if one wants to prove the numerical existence property only for a fragment of the theory. For this purpose different hierarchical classifications of formulae are presented and analyzed, namely the well-known Σ^0_n- and Π^0_n-hierarchies, as well as a classification that fits better into the context of intuitionistic logic, the so-called Φ_n-hierarchy which is due to [Bur00]. To some extent, the optimality of the obtained results is discussed. Another subsection of section 4 discusses the relation of the disjunction and the numerical existence properties of Heyting arithmetic. In particular, an interesting result which is due to [Lei85] is presented. It states that Heyting arithmetic proves the following: "If Heyting arithmetic obeys the disjunction property for disjunctions of Σ^0_2-sentences then it has the numerical existence property". This consequence of the theorem from section 3 follows by the soundness of the q-realizability predicate. The q-realizability predicate is introduced and its soundness is (succinctly) proven.
In section 5 a different version of the result of section 3 is shown. If instead of the disjunction property of a recursively enumerable extension of (intuitionistic) arithmetic T we assume that for a formula P and all sentences A and B the statement $\mathrm{T} \vdash P \rightarrow A \vee B$ implies that either $\mathrm{T} \vdash P \rightarrow A$ or $\mathrm{T} \vdash P \rightarrow B$ holds, then T also fulfills the respective version (that is, with P as a premise) of the numerical existence property.
Interestingly, the proof of the theorem given in section 3 can be formalized in HA_0, which is the same as Heyting arithmetic, except that the induction schema is restricted to quantifier-free formulae. This allows Friedman to easily prove a further theorem in [Fri75], which states that if a recursively enumerable extension of (intuitionistic) arithmetic proves its own disjunction property, then it also proves its own

inconsistency. Friedman's rather succinct proof of this theorem is worked out in detail in section 6, the final section of this thesis. Basically, the theorem of section 6 is a consequence of the theorem of section 3 and Löb's theorem, which is also proven in section 6.

2 Preliminaries

In the following HA will denote the usual formulation of *Heyting arithmetic*, that is, intuitionistic first-order arithmetic. HA is based on one-sorted intuitionistic predicate calculus with identity. Its language contains a constant 0, distinct numerical variables w, x, y, z (also $w', x', y', z', x_0, x_1, x_2, \ldots$), a unary function symbol S denoting the successor function, function symbols for all primitive recursive functions.

Let HA_0 be the same as HA, except that the induction schema is restricted to quantifier-free formulae.

Definition 2.1. A theory T is called an *extension of arithmetic*, if its axioms include those of HA_0.

Furthermore, let PA denote the usual formulation of Peano arithmetic, which is HA plus the reductio-ad-absurdum rule, or alternatively HA plus the law-of-excluded-middle schema.

2.1 Some properties of Heyting arithmetic

The following lemmata will be useful throughout this thesis.

Lemma 2.2. $\mathrm{HA}_0 \vdash x = 0 \lor x \neq 0$.

Proof. This can be easily proven by the use of the induction scheme for quantifier-free formulae IS_0 of HA_0.

$$\dfrac{\dfrac{0=0}{0=0\lor 0\neq 0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x=0\lor x\neq 0]^{(1)}}{\mathrm{S}(x)\neq 0}}{\mathrm{S}(x)=0\lor \mathrm{S}(x)\neq 0}}{x=0\lor x\neq 0\to \mathrm{S}(x)=0\lor \mathrm{S}(x)\neq 0}\,(\to\text{-I})^{(1)}}{\forall x\,(x=0\lor x\neq 0\to \mathrm{S}(x)=0\lor \mathrm{S}(x)\neq 0)}}{\forall x\,(x=0\lor x\neq 0)}\;\mathrm{IS}_0$$

□

Lemma 2.3. *Let A, B be formulae. It holds that*

$$\mathrm{HA}_0 \vdash (A\lor B) \leftrightarrow \exists x\,((x=0\to A)\land(x\neq 0\to B))\,.$$

Proof. $\mathrm{HA}_0 \vdash (A\lor B) \to \exists x\,((x=0\to A)\land(x\neq 0\to B))$ holds by the following derivation.

$$\dfrac{A\lor B \qquad \dfrac{\dfrac{\dfrac{[A]}{0=0\to A} \quad \dfrac{\dfrac{0=0}{0\neq 0\to\bot}}{0\neq 0\to B}}{(0=0\to A)\land(0\neq 0\to B)}}{\exists x\,((x=0\to A)\land(x\neq 0\to B))} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathrm{S}(0)\neq 0}{\mathrm{S}(0)=0\to\bot}}{\mathrm{S}(0)=0\to A} \quad \dfrac{[B]}{\mathrm{S}(0)\neq 0\to B}}{(\mathrm{S}(0)=0\to A)\land(\mathrm{S}(0)\neq 0\to B)}}{\exists x\,((x=0\to A)\land(x\neq 0\to B))}}{\exists x\,((x=0\to A)\land(x\neq 0\to B))}\;(\lor\text{-E})$$

The converse can be shown as follows.

$$\dfrac{\dfrac{\dfrac{\text{lemma 2.2}}{x=0\lor x\neq 0} \qquad \dfrac{\dfrac{[(x=0\to A)\land(x\neq 0\to B)]^{(2)} \quad [x=0]^{(1)}}{A}}{A\lor B} \qquad \dfrac{\dfrac{[(x=0\to A)\land(x\neq 0\to B)]^{(2)} \quad [x\neq 0]^{(1)}}{B}}{A\lor B}}{A\lor B}\;(\lor\text{-E})^{(1)}}{\exists x\,((x=0\to A)\land(x\neq 0\to B))\to(A\lor B)}\;(\exists\text{-E})^{(2)},\,(\to\text{-I})$$

□

2.2 Some important primitive recursive functions

The following results are well-known. Hence I refer to widely-used literature instead of proving these results here.

Lemma 2.4. *Let* T *be a recursively enumerable extension of arithmetic.*

(i) *There is an injective function* $\ulcorner\cdot\urcorner$ *from the set of all formulae of* T *into* $\mathbb{N}_0$. *It is called the* Gödel numbering.

(ii) *There are primitive recursive functions* $\mathrm{prf}_T, \mathrm{neg}, \mathrm{sub}, \mathrm{imp} : \mathbb{N}_0 \to \mathbb{N}_0$ *such that for any formulae A, B,* $C = C(x)$ *in the language of* T*, where C is a formula with no free variable other than x, it holds that*

(1) $\mathrm{T} \vdash A$ *if and only if there is a number n, such that* $\mathrm{prf}_T(n, \ulcorner A \urcorner) = 0$*,*

(2) $\mathrm{neg}(\ulcorner A \urcorner) = \ulcorner \neg A \urcorner$*,*

(3) $\mathrm{sub}(\ulcorner C(x) \urcorner) = \ulcorner C\left(\overline{\ulcorner C(x) \urcorner}\right) \urcorner$*,*

(4) $\mathrm{imp}(\ulcorner A \urcorner, \ulcorner B \urcorner) = \ulcorner A \to B \urcorner$.

Proof. See section 3 in [Smo82]. □

Lemma 2.5. *For each primitive recursive function f there is a primitive recursive function symbol* $\overline{f}$*, such that* $f(n) = m$ *if and only if* $\mathrm{HA}_0 \vdash \overline{f}(\overline{n}) = \overline{m}$.

Proof. See section 3 in [Smo82]. □

For a given primitive recursive function f, from here on let $\overline{f}$ denote the corresponding function symbol of T. That is, $\overline{\mathrm{prf}_T}, \overline{\mathrm{neg}}, \overline{\mathrm{sub}}, \overline{\mathrm{imp}}$ are the function symbols for $\mathrm{prf}_T, \mathrm{neg}, \mathrm{sub}, \mathrm{imp}$ respectively, fulfilling the property indicated in lemma 2.5.

Definition 2.6 (Bounded Minimalization)**.** For a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ and a $y \in \mathbb{N}_0$ let $\mu_{x \le y}\left(f(x) = 0\right)$ stand for the least $x \in \mathbb{N}_0$, such that $x \le y$ and $f(x) = 0$, if such a number exists. Otherwise let $\mu_{x \le y}\left(f(x) = 0\right) := 0$.

Lemma 2.7 (Bounded Minimalization)**.** *Let* $f : \mathbb{N}_0 \to \mathbb{N}_0$ *be a primitive recursive function and let* $g : \mathbb{N}_0 \to \mathbb{N}_0$ *be defined by* $g(y) := \mu_{x \le y}\left(f(x) = 0\right)$. *Then g is a primitive recursive function and it holds*

(i) $\mathrm{HA}_0 \vdash \overline{g}(y) \le y$*,*

(ii) $\mathrm{HA}_0 \vdash \overline{f}(y) = 0 \to \overline{f}\left(\overline{g}(y)\right) = 0$*,*

(iii) $\mathrm{HA}_0 \vdash z < \overline{g}(y) \to \overline{f}(z) \ne 0$*,*

(iv) $\mathrm{HA}_0 \vdash \overline{g}(y) \ne 0 \to \overline{f}\left(\overline{g}(y)\right) = 0$*,*

(v) $\mathrm{HA}_0 \vdash y \le z \to \overline{g}(y) \le \overline{g}(z)$.

Proof. See section 2.9 in [Goo57]. □

2.3 Provability predicate

For a recursively enumerable extension of arithmetic T one can define a formula with a single free variable, which given (the numeral of) the Gödel number of a formula A as input expresses that A is provable in T. The *provability predicate* and its properties are presented below.
Later in this subsection the concept of a "Gödel number containing a free variable" is introduced. It can be inserted into the provability predicate, which results in a formula with a single free variable. Then some interesting properties of this formula, which are essential for the main proof of section 6, are presented.

Definition 2.8 (Provability Predicate)**.** Let T be a recursively enumerable extension of arithmetic. Define

$$\mathrm{Pr}_T\left(y\right) :\equiv \exists x \left(\overline{\mathrm{prf}}_T\left(x, y\right) = 0\right).$$

$\mathrm{Pr}_T\left(y\right)$ is called *provability predicate* of T.

The provability predicate $\mathrm{Pr}_T\left(y\right)$ fulfills the so-called *derivability conditions of Hilbert and Bernays*. Löb's version of these important properties of $\mathrm{Pr}_T\left(y\right)$ is given in the following lemma.

Lemma 2.9 (Derivability Conditions)**.** *Let* T *be a recursively enumerable extension of arithmetic. Let* A, B *be formulae.*
$\mathrm{Pr}_T\left(y\right)$ *has the following properties*

(D1) : $\mathrm{T} \vdash A$ *implies* $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right)$,

(D2) : $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \urcorner}\right)$,

(D3) : $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \wedge \mathrm{Pr}_T\left(\overline{\ulcorner A \rightarrow B \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right)$.

Proof. In section 3 of [Smo82] the derivability conditions are formulated and (sketchily) proven for all sentences A and B. Although Smorynski states the assertion only for sentences, the proof that he gives works for open formulae as well. Furthermore, a very detailed proof of the derivability conditions for all formulae (as required here) is given in section II.6 of [Tou03]. Also it could be noticed that both, Smorynski and Tourlakis, assume T to be an extension of Peano arithmetic in their proofs. Nevertheless both proofs work for extensions of HA_0 as well. □

Some further properties of $\mathrm{Pr}_T\left(y\right)$, which will be used later in this thesis, are given in the following lemma. Of course more properties of $\mathrm{Pr}_T\left(y\right)$, that are similar to those presented below, can be proven in the same manner as the presented.

Lemma 2.10. *Let* T *be a recursively enumerable extension of arithmetic. Let* A*,* B *be arbitrary formulae and let* $C(y)$ *be a formula with a free variable* y*. Then it holds*

(i) $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A \vee B \urcorner}\right)$,

(ii) $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \wedge \mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A \wedge B \urcorner}\right)$,

(iii) $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \rightarrow \bot \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A \rightarrow B \urcorner}\right)$,

(iv) $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner C(t) \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \exists y C(y) \urcorner}\right)$, *where* t *is a term, which is free for* y *in* C.

Proof. (i) The assertion follows from $\mathrm{T} \vdash A \rightarrow A \vee B$ by (D1) and (D3).

(ii) It holds $\mathrm{T} \vdash A \rightarrow (B \rightarrow A \wedge B)$. By (D1) and (D3) it follows

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner (B \rightarrow A \wedge B) \urcorner}\right).$$

Furthermore, by (D3) holds

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner (B \rightarrow A \wedge B) \urcorner}\right) \wedge \mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A \wedge B \urcorner}\right).$$

The two statements yield the assertion.

(iii) It holds $\mathrm{T} \vdash (A \rightarrow \bot) \rightarrow (A \rightarrow B)$. As before (D1) and (D3) imply the assertion.

(iv) The assertion follows from $\mathrm{T} \vdash C(t) \rightarrow \exists y C(y)$ by (D1) and (D3). □

Assume $A(y)$ to be a formula of a recursively enumerable extension of arithmetic T with no free variable other than y. By looking at how exactly the Gödel numbering is carried out (see for example section 3 in [Smo82]) it becomes clear that a function $f: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ can be defined, such that $f(n) = \ulcorner A(n) \urcorner$ for all $n \in \mathbb{N}_0$. That is, informally $f(x)$ can be described as the "Gödel number" of $A(x)$, but where x is regarded as a free variable even though it is "inside" the Gödel numbering $\ulcorner\urcorner$. Additionally it turns out that such functions are primitive recursive.
The following lemma treats this observation in a formal manner.

Lemma 2.11 (The notation $\overline{\ulcorner A(\dot{y}) \urcorner}$)**.** *Let* T *be a recursively enumerable extension of arithmetic.*

(i) *There is a primitive recursive function* $\mathrm{s}(x, y)$*, such that for every formula* $A(z)$ *with only* z *free and every number* n *it holds*

$$\mathrm{T} \vdash \bar{\mathrm{s}}\left(\overline{\ulcorner A(z) \urcorner}, \bar{n}\right) = \overline{\ulcorner A(\bar{n}) \urcorner}.$$

(ii) *Throughout this thesis the term* $\bar{\mathrm{s}}\left(\overline{\ulcorner A(z) \urcorner}, y\right)$*, which has* y *as a single free variable, is abbreviated by* $\overline{\ulcorner A(\dot{y}) \urcorner}$.

Proof. See section 3.2.2 in [Smo82], or pages 296-300 in [Tou03] for a more detailed explanation. □

Thus, assuming $A(y)$ to be a formula of a recursively enumerable extension of arithmetic T with no free variable other than y, also $\mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{y}) \urcorner}\right)$ is a formula with a single free variable y. The rest of this subsection presents some important properties of such formulae.

Lemma 2.12. *Let* T *be a recursively enumerable extension of arithmetic. Let* f *be a 1-ary primitive recursive function. Then*

$$\mathrm{T} \vdash \bar{f}(x) = 0 \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \bar{f}(\dot{x}) = 0 \urcorner}\right).$$

In particular for any formula A *it holds*

$$\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(x, \overline{\ulcorner A \urcorner}\right) = 0 \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{x}, \overline{\ulcorner A \urcorner}\right) = 0 \urcorner}\right).$$

Proof. Section 3.2.5 in [Smo82] sketches a proof which proceeds by induction on the number of steps needed to generate f. For a detailed version of the proof see section II.6.34 in [Tou03]. □

Lemma 2.13 (Free Variable Versions of (D1) and (D3))**.** *Let* T *be a recursively enumerable extension of arithmetic. Let* $A(x), B(x)$ *be formulae with only* x *free. It holds that*

(D1*) : $\mathrm{T} \vdash A(x)$ *implies* $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \urcorner}\right)$,

(D3*) : $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \urcorner}\right) \wedge \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \rightarrow B(\dot{x}) \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner B(\dot{x}) \urcorner}\right)$.

Proof. For (D1*) see [Tou03] II.6.30 and for (D3*) see [Tou03] II.6.31. □

Now we can reformulate lemma 2.10 to a free variable version as well.

Corollary 2.14. *Let* T *be a recursively enumerable extension of arithmetic. Let* $A(x)$, $B(x)$ *be formulae with only* x *free. Then it holds that*

(i) $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \vee B(\dot{x}) \urcorner}\right)$,

(ii) $T \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \urcorner}\right) \wedge \mathrm{Pr}_T\left(\overline{\ulcorner B(\dot{x}) \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \wedge B(\dot{x}) \urcorner}\right)$,

(iii) $T \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \rightarrow \bot \urcorner}\right) \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A(\dot{x}) \rightarrow B(\dot{x}) \urcorner}\right)$.

Proof. Just substitute (D1*) and (D3*) for (D1) and (D3) respectively in the proof of lemma 2.10 □

3 The numerical existence property is equivalent to the disjunction property

Let us begin with a definition of the properties in question.

Definition 3.1 (Disjunction Property, Numerical Existence Property)**.** An extension of arithmetic T is said to have the *disjunction property* (abbr.: DP), if for all sentences A and B the statement $\mathrm{T} \vdash A \vee B$ implies that either $\mathrm{T} \vdash A$ or $\mathrm{T} \vdash B$ holds.
T is said to have the *numerical existence property* (abbr.: NEP), if for any formula $A(x)$ with no free variable other than x the statement $\mathrm{T} \vdash \exists x A(x)$ implies that there is a number n, such that $T \vdash A(\overline{n})$ holds.

It is easy to see that the numerical existence property of an extension of arithmetic implies its disjunction property (just apply lemma 2.3). More surprisingly it turns out that every recursively enumerable extension of arithmetic which obeys the disjunction property, also obeys the numerical existence property. A proof of this assumption was given in the article [Fri75]. The objective of this section is to formalize the rather short proof given there in a more detailed way.

Friedman's proof is based on several self referential sentences that stated in colloquial language are similar to:

> *"Either the Gödel number of any proof of this sentence is an upper bound for some n such that $P(\overline{n})$ is provable, or it is an upper bound for the Gödel number of some proof of the negation of this sentence."*

For now, let us call this sentence A. Using the disjunction property one could derive from a proof of A either an n such that $\mathrm{T} \vdash P(\overline{n})$ or a proof of $\neg A$, because either n or the Gödel number of a proof of $\neg A$ would be bounded by the Gödel number of the proof of A. Otherwise, from a proof of $\neg A$ one could derive a proof of A, because either the Gödel number of any proof of A would be such as required in the statement of A (that is, if each number, which is smaller than the Gödel number of the proof of $\neg A$, is not a Gödel number of a proof of A), or a proof of A would be known (that is, if there exists a proof of A with a Gödel number smaller than the number of the proof of $\neg A$). Thus, in order to find an n such that $\mathrm{T} \vdash P(\overline{n})$, it would be sufficient to establish $\mathrm{T} \vdash \exists y P(y) \to A \vee \neg A$. Unfortunately this turns out to be false in general. Therefore the proof presented here is not as simple as the concept outlined just now, which nevertheless expresses the basic idea of the proof.
The following lemma will be used in the proof.

Lemma 3.2. *Let* T *be an extension of arithmetic. Let $C(y)$ be a formula with a free variable y. Then for an arbitrary number j holds*

(i) $\mathrm{T} \vdash y \leq \overline{j} \wedge C(y) \to \bigvee_{i \leq j} C(\overline{i})$,

(ii) $\mathrm{T} \vdash \bigwedge_{i<j} C(\overline{i}) \to \left(y < \overline{j} \to C(y)\right)$.

Proof. (i) Apply induction on j.
Clearly, $\mathrm{T} \vdash y \leq 0 \wedge C(y) \to C(0)$. Further, if (i) holds for some j then it holds for $\mathrm{S}(j)$ as well:

$$\dfrac{\dfrac{\dfrac{\dfrac{y \leq \mathrm{S}(\overline{j}) \wedge C(y)}{C(y)} \qquad \left[y = \mathrm{S}(\overline{j})\right]}{C(\mathrm{S}(\overline{j}))}}{\bigvee_{i \leq \mathrm{S}(j)} C(\overline{i})} \qquad \dfrac{\dfrac{\dfrac{y \leq \mathrm{S}(\overline{j}) \wedge C(y)}{C(y)} \qquad \left[y \leq \overline{j}\right]}{\bigvee_{i \leq j} C(\overline{i})}\ \text{(I.H.)}}{\bigvee_{i \leq \mathrm{S}(j)} C(\overline{i})}}{\bigvee_{i \leq \mathrm{S}(j)} C(\overline{i})}\ (\vee\text{-E})$$

where $T \vdash y \leq S(\bar{j}) \leftrightarrow y \leq \bar{j} \vee y = S(\bar{j})$ was used to apply the ($\vee$-E)-rule.

(ii) Again apply induction on j.
If $j = 0$ it is to show that $T \vdash y < 0 \rightarrow C(y)$, which follows at once from $T \vdash y < 0 \rightarrow \bot$. And if (ii) holds for some j, then it holds for $S(j)$ as well:

$$\dfrac{\dfrac{\dfrac{\bigwedge_{i<S(j)} C(\bar{i})}{C(\bar{j})} \quad [y = \bar{j}]}{C(y)} \qquad \dfrac{[y < \bar{j}] \quad \dfrac{\dfrac{\bigwedge_{i<S(j)} C(\bar{i})}{\bigwedge_{i<j} C(\bar{i})}}{y < \bar{j} \rightarrow C(y)}\text{(I.H.)}}{C(y)}}{y < S(\bar{j}) \rightarrow C(y)}\,(\vee\text{-E})$$

where $T \vdash y < S(\bar{j}) \leftrightarrow y < \bar{j} \vee y = \bar{j}$ was used for the ($\vee$-E)-rule. □

Theorem 3.3 (DP $\Leftrightarrow$ NEP. Due to [Fri75]). *Let* T *be a recursively enumerable extension of arithmetic. Then* T *obeys the numerical existence property if and only if* T *obeys the disjunction property.*

Proof. Let T be a recursively enumerable extension of arithmetic, which obeys the disjunction property. Let $P(y)$ be a formula with no free variable other than y. Let $A_P(x)$ denote the formula

$$\exists y\left(\left(\overline{\mathrm{prf}}_T(y, \overline{\mathrm{neg}}(x)) = 0 \vee P(y)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T(z, x) = 0 \rightarrow y \leq z\right)\right).$$

Let $k := \ulcorner A_P\left(\overline{\mathrm{sub}}(x)\right)\urcorner$. Then $\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\urcorner = \mathrm{sub}(k)$.
Let $Q(y)$ denote the formula $\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\bar{k}\right)\right) = 0$. Let $A_Q(x)$ be the same formula as $A_P(x)$, except that Q is substituted for P, i.e.

$$\exists y\left(\left(\overline{\mathrm{prf}}_T(y, \overline{\mathrm{neg}}(x)) = 0 \vee \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\bar{k}\right)\right) = 0\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T(z, x) = 0 \rightarrow y \leq z\right)\right).$$

As above, choose a number l such that $\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner = \mathrm{sub}(l)$.

Claim 1: $T \vdash \exists y P(y) \rightarrow A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right) \vee A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right) \vee \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$.

After this claim is shown, the disjunction property can be applied to prove the theorem.

Let f, g, h be defined by

$$\begin{aligned} f(y) &:= \mu_{x \leq y}\left(\mathrm{prf}_T(x, \mathrm{sub}(k)) = 0\right), \\ g(y) &:= \mu_{x \leq y}\left(\mathrm{prf}_T(x, \mathrm{sub}(l)) = 0\right), \\ h(y) &:= \mu_{x \leq y}\left(\mathrm{prf}_T(x, \mathrm{sub}(k)) \cdot \mathrm{prf}_T(x, \mathrm{neg}(\mathrm{sub}(l))) = 0\right). \end{aligned}$$

The five derivations, that are shown below, will combined establish the claim.

Derivation 1:

$$\dfrac{\dfrac{P(y) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}{\overline{f}(y)=0}\,(2.7(\mathrm{iv})) \qquad \dfrac{[z\leq y]}{\overline{f}(z)\leq\overline{f}(y)}\,(2.7(\mathrm{v}))}{\overline{f}(z)=\overline{f}(y)} \qquad \overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}{\overline{\mathrm{prf}}_T\left(\overline{f}(z),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}\,(2.7(\mathrm{ii}))}{z\leq y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y<z}}{\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z}}{\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)}}{\left(P(y)\vee\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)}}{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)}$$

Derivation 2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0}{\overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0}}{\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0}\,(2.7(\mathrm{ii}))}{\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0} \qquad \dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\neq 0}{\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)=0} \\ \vdots \\ \text{(analog to derivation 1)} \\ \vdots}{\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow\overline{h}\left(\overline{f}(y)\right)\leq z\right)}}{\left(\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow\overline{h}\left(\overline{f}(y)\right)\leq z\right)}}{A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)}$$

Derivation 3:

$$\dfrac{\dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(\overline{f}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0}{\vdots} \\ \text{(same as in derivation 2)} \\ \vdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{lemma 2.7(iii)}}{z<\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\neq 0} \qquad \overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)=\overline{h}\left(\overline{f}(y)\right)}{z<\overline{h}\left(\overline{f}(y)\right)\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow\overline{h}\left(\overline{f}(y)\right)\leq z}}{\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow\overline{h}\left(\overline{f}(y)\right)\leq z\right)}}{\left(\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(\overline{h}\left(\overline{f}(y)\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow\overline{h}\left(\overline{f}(y)\right)\leq z\right)}}{A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)}$$

Derivation 4:

$$\dfrac{\dfrac{\left[A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right]^{(2)} \qquad \dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0 \qquad [(*)]^{(1)}}{w\leq\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)} \qquad \dfrac{\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)<\overline{h}\left(\overline{f}(y)\right) \qquad \dfrac{[(*)]^{(1)}}{\text{(Derivation 5)}}\;\overline{h}\left(\overline{f}(y)\right)\leq w}{\overline{g}\left(\overline{h}\left(\overline{f}(y)\right)\right)<w}}{\bot}}{\bot}\,(\exists\text{-E})^1}{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)}\,(\rightarrow\text{-I})^2$$

where $(*)$ denotes the formula $\left(\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\right)\wedge\forall x\left(\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\rightarrow w\leq x\right)$.

Derivation 5:

$$\dfrac{\dfrac{\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0}{\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0} \quad \dfrac{\dfrac{\dfrac{\overset{\text{lemma 2.7(iii)}}{\overline{\quad}}}{x<\overline{h}\left(\overline{f}\left(y\right)\right)\rightarrow\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\rightarrow\overline{h}\left(\overline{f}\left(y\right)\right)\leq x}}{\dfrac{\forall x\left(\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(x,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\rightarrow\overline{h}\left(\overline{f}\left(y\right)\right)\leq x\right)}{\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\overline{\mathrm{prf}}_T\left(w,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right)=0\rightarrow\overline{h}\left(\overline{f}\left(y\right)\right)\leq w}}}{\overline{h}\left(\overline{f}\left(y\right)\right)\leq w}$$

Suppose $\mathrm{T}\vdash\exists y P\left(y\right)$. Towards an application of the (∃-E)-rule let $P\left(y\right)$.
If $\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0$ then by derivation 1 follows $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$.
If $\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0$ and $\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\neq 0$ then derivation 2 yields $A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$.
If $\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0$, $\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0$ and $\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)=\overline{h}\left(\overline{f}\left(y\right)\right)$ then derivation 3 implies $A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$.
If $\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0$, $\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0$ and $\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)<\overline{h}\left(\overline{f}\left(y\right)\right)$ then by derivation 4 follows $\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$. Now using

$$\begin{aligned}
&\mathrm{T}\vdash\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0,\\
&\mathrm{T}\vdash\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)=0\vee\overline{\mathrm{prf}}_T\left(\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right),\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\neq 0,\\
&\mathrm{T}\vdash\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)\leq\overline{h}\left(\overline{f}\left(y\right)\right)\ (\text{by lemma 2.7 (i)}),\\
&\mathrm{T}\vdash\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)\leq\overline{h}\left(\overline{f}\left(y\right)\right)\leftrightarrow\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)=\overline{h}\left(\overline{f}\left(y\right)\right)\vee\overline{g}\left(\overline{h}\left(\overline{f}\left(y\right)\right)\right)<\overline{h}\left(\overline{f}\left(y\right)\right)
\end{aligned}$$

three applications of the (∨-E)-rule yield $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\vee A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\vee\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$.
Claim 1 follows by the (∃-E)-rule.

Thus from $\mathrm{T}\vdash\exists y P\left(y\right)$ by the disjunction property follows that either $\mathrm{T}\vdash A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ or $\mathrm{T}\vdash A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$ or $\mathrm{T}\vdash\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$ holds. In the following each of this three cases will be considered separately. So assume $\mathrm{T}\vdash\exists y P\left(y\right)$.

First case: If $\mathrm{T}\vdash A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$, then $\mathrm{T}\vdash P\left(\overline{n}\right)$ for some n.

Assume $\mathrm{T}\vdash A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$. From $\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\urcorner=\mathrm{sub}\left(k\right)$ follows that there exists a number m, such that $\mathrm{prf}_T\left(m,\mathrm{sub}\left(k\right)\right)=0$ and consequently $\mathrm{T}\vdash\overline{\mathrm{prf}}_T\left(\overline{m},\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0$. Fix m.
We have,

$$\dfrac{\dfrac{\dfrac{\left(\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\vee P\left(y\right)\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)}{\overline{\mathrm{prf}}_T\left(\overline{m},\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq\overline{m}}\ (\wedge\text{-E}),\ (\forall\text{-E})}{y\leq\overline{m}}\ (\rightarrow\text{-E})}{\exists y\left(y\leq\overline{m}\wedge\left(\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\vee P\left(y\right)\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)\right)}\ (\wedge\text{-I}),\ (\exists\text{-I})$$

Hence by the (∃-E)-rule applied to $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ holds

$$\mathrm{T}\vdash\exists y\left(y\leq\overline{m}\wedge\left(\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\vee P\left(y\right)\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)\right).$$

It follows,

$$\frac{\dfrac{\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge \left(\overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0 \vee P\left(y\right)\right)\right)}{\mathrm{T} \vdash \exists y \left(\left(y \le \overline{m} \wedge \overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0\right) \vee \left(y \le \overline{m} \wedge P\left(y\right)\right)\right)}}{\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge \overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0\right) \vee \exists y \left(y \le \overline{m} \wedge P\left(y\right)\right)}$$

Thus either $\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge P\left(y\right)\right)$ or $\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge \overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0\right)$ holds, since T obeys the disjunction property.
First subcase: Suppose $\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge P\left(y\right)\right)$.
By lemma 3.2 it holds $\mathrm{T} \vdash y \le \overline{m} \wedge P\left(y\right) \to \bigvee_{i \le m} P\left(\overline{i}\right)$. So, applying the (∃-E)-rule to the assumption one gets $\mathrm{T} \vdash \bigvee_{i \le m} P\left(\overline{i}\right)$, and consequently, $\mathrm{T} \vdash P\left(\overline{i}\right)$ for some $i \le m$, since T has the disjunction property.
Second subcase: Suppose $\mathrm{T} \vdash \exists y \left(y \le \overline{m} \wedge \overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0\right)$.
By lemma 3.2 we have

$$\mathrm{T} \vdash y \le \overline{m} \wedge \overline{\mathrm{prf}_T} \left(y, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0 \to \bigvee_{i \le m} \overline{\mathrm{prf}_T} \left(\overline{i}, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0.$$

Hence, by the (∃-E)-rule applied to the assumption it follows

$$\mathrm{T} \vdash \bigvee_{i \le m} \overline{\mathrm{prf}_T} \left(\overline{i}, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0,$$

and consequently by the disjunction property for some $i \le m$ holds

$$\mathrm{T} \vdash \overline{\mathrm{prf}_T} \left(\overline{i}, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)\right) = 0.$$

But then $\mathrm{T} \vdash \neg A_P \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)$, because $\ulcorner \neg A_P \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right) \urcorner = \mathrm{neg}\left(\ulcorner A_P \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right) \urcorner\right) = \mathrm{neg}\left(\mathrm{sub}\left(k\right)\right)$. So T is inconsistent.
Hence in both subcases, $\mathrm{T} \vdash P\left(\overline{n}\right)$ holds for some number n.

Second case: If $\mathrm{T} \vdash A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)$, then $\mathrm{T} \vdash P\left(\overline{n}\right)$ for some n.

The same argumentation as in the first case, except that Q is substituted for P and l for k, yields $\mathrm{T} \vdash Q\left(\overline{n}\right)$ for some number n; that is, $\mathrm{T} \vdash \overline{\mathrm{prf}_T} \left(\overline{n}, \overline{\mathrm{sub}} \left(\overline{k}\right)\right) = 0$. Since $\ulcorner A_P \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right) \urcorner = \mathrm{sub}\left(k\right)$ it holds $\mathrm{T} \vdash A_P \left(\overline{\mathrm{sub}} \left(\overline{k}\right)\right)$. Hence the assumption follows by the first case.

Third case: If $\mathrm{T} \vdash \neg A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)$, then T is inconsistent.

Assume $\mathrm{T} \vdash \neg A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)$. Then $\mathrm{prf}_T \left(m, \mathrm{neg}\left(\mathrm{sub}\left(l\right)\right)\right) = 0$ for some m, because $\ulcorner \neg A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right) \urcorner = \mathrm{neg}\left(\ulcorner A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right) \urcorner\right) = \mathrm{neg}\left(\mathrm{sub}\left(l\right)\right)$. Fix m.
Suppose $\mathrm{prf}_T \left(i, \mathrm{sub}\left(l\right)\right) = 0$ for some $i < m$. Then $\mathrm{T} \vdash A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)$, because $\mathrm{sub}\left(l\right) = \ulcorner A_Q \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right) \urcorner$. Hence T is inconsistent in this case.
Otherwise, suppose $\mathrm{prf}_T \left(i, \mathrm{sub}\left(l\right)\right) \ne 0$ for every $i < m$. Then $\mathrm{T} \vdash \bigwedge_{i<m} \overline{\mathrm{prf}_T} \left(\overline{i}, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) \ne 0$. By lemma 3.2 we have $\mathrm{T} \vdash \bigwedge_{i<m} \overline{\mathrm{prf}_T} \left(\overline{i}, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) \ne 0 \to \left(x < \overline{m} \to \overline{\mathrm{prf}_T} \left(x, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) \ne 0\right)$. Hence it follows that $\mathrm{T} \vdash x < \overline{m} \to \overline{\mathrm{prf}_T} \left(x, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) \ne 0$ holds. By contraposition and the (∀-I)-rule this implies $\mathrm{T} \vdash \forall x \left(\overline{\mathrm{prf}_T} \left(x, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) = 0 \to \overline{m} \le x\right)$.
Furthermore it holds $\mathrm{T} \vdash \overline{\mathrm{prf}_T} \left(\overline{m}, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)\right) = 0$ as observed above. Hence

$$\mathrm{T} \vdash \left(\overline{\mathrm{prf}_T} \left(\overline{m}, \overline{\mathrm{neg}} \left(\overline{\mathrm{sub}} \left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\overline{m}\right)\right) \wedge \forall x \left(\overline{\mathrm{prf}_T} \left(x, \overline{\mathrm{sub}} \left(\overline{l}\right)\right) = 0 \to \overline{m} \le x\right).$$

And therefore by the (∃-I)-rule it follows $\mathrm{T} \vdash A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$. Thus T is inconsistent.

So it was shown that $\mathrm{T} \vdash \exists y P\left(y\right) \rightarrow A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right) \vee A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right) \vee \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$ holds, and that, after the application of the disjunction property to this formula, in each of the three cases $\mathrm{T} \vdash P\left(\overline{n}\right)$ holds for some number n. Thus, if T obeys the disjunction property, then it also obeys the numerical existence property.

The converse follows by means of lemma 2.3. □

Remark 3.4. Notice that in the proof of theorem 3.3 instead of the formula $A_P\left(x\right)$ one could also use the shorter formula

$$\exists y\left(P\left(y\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, x\right) = 0 \rightarrow y \leq z\right)\right)$$

That is, one disjunction in the former definition of $A_P\left(x\right)$ can be dropped. In fact, the purpose of the dropped part was to enable that $\mathrm{T} \vdash \neg A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)$ would imply $\mathrm{T} \vdash A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)$. But such an implication is only needed for the formula $A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$ for the third case in the proof. That is also the reason why the formula $A_Q\left(x\right)$ cannot be used in the proof in such a simplified form. Hence such a simplification of $A_P\left(x\right)$ would actually make the proof longer, because of the loss of the previous resemblance between $A_P\left(x\right)$ and $A_Q\left(x\right)$.

4 Corollaries

After theorem 3.3 is established, certain questions which are related to the theorem or its proof arise. Hence the aim of this section is to present some corollaries of theorem 3.3 and its proof.
For instance it turns out that to establish the numerical existence property not for all formulae with a single free variable, but only for a smaller class, the disjunction property is also not required to hold for all disjunctions of sentences, but only for a smaller class. A detailed treatment of this matter for an arbitrary recursively enumerable extension of arithmetic T is given below in subsection 4.2. Interestingly, the general requirements sometimes can be weakened as shown by an analysis of HA in the same subsection.
But first let us consider whether an algorithm for the witness number can be deduced from the proof of theorem 3.3.

4.1 On the witness number

A question which arises naturally is, whether the proof of theorem 3.3 gives a bound for the witness number, or maybe even an algorithm to compute the bound or the number. Several corollaries of the proof of theorem 3.3 address this question in the following.
Throughout this subsection let T, $P(y)$, $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ be defined as in the proof of theorem 3.3.

Corollary 4.1 (On the upper bound for the witness number)**.** *If* T *is consistent, then* $\mathrm{T} \vdash \overline{\text{prf}}_T\left(\overline{m}, \overline{\ulcorner A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner}\right) = 0$ *for some* m *and* $\mathrm{T} \vdash P\left(\overline{n}\right)$ *for some* $n \leq m$.

Proof. Consider the case distinction from the proof of theorem 3.3.
(*Third case*) If $\neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ is provable, then T is inconsistent.
(*Second case*) If $A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ is provable and m_Q is a number fulfilling

$$\mathrm{T} \vdash \overline{\text{prf}}_T\left(\overline{m_Q}, \overline{\ulcorner A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner}\right) = 0,$$

then $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ is provable and there is a number $m_P \leq m_Q$ fulfilling

$$\mathrm{T} \vdash \overline{\text{prf}}_T\left(\overline{m_P}, \overline{\ulcorner A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner}\right) = 0.$$

(*First case*) If $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ is provable and m_P is a number fulfilling

$$\mathrm{T} \vdash \overline{\text{prf}}_T\left(\overline{m_P}, \overline{\ulcorner A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner}\right) = 0,$$

then there is a number $n \leq m_P$, such that $\mathrm{T} \vdash P\left(\overline{n}\right)$ holds. □

Corollary 4.2 (An algorithm for the witness number)**.** *Assume that an algorithm, that transforms the code of a proof of a closed disjunction* $A \vee B$ *into the code of a proof of either* A *or* B*, is known. Then one can derive from the proof of theorem 3.3 an algorithm, which out of the code of a proof of* $\exists y P(y)$ *computes a number* n *such that* $\mathrm{T} \vdash P(\overline{n})$.

Proof. This easily becomes clear by going through the proof of theorem 3.3.
The proof consists of several case distinctions. Whenever a case of a case distinction does not imply inconsistency of T, it actually gives a suitable n. In fact, such a number n is found utilizing besides the rules and axioms of T only the disjunction property and the sentence $\exists y P(y)$, which is assumed to be already proven. Clearly, if a syntactical proof of a statement in T is given, then this proof can be encoded by some algorithm. Thus using the disjunction-property-algorithm from the assumption, one can build an algorithm which computes a witness number n out of the code of a proof of $\exists y P(y)$. Otherwise, whenever a case of a case distinction implies inconsistency of T, we may set the witness number n to 0. □

Corollary 4.3 (An algorithm for the upper bound of the witness number)**.** *Assume there is an algorithm that produces out of the code of a proof of a closed disjunction $A \vee B$ an upper bound for the code of a proof of either A or B. Then one can derive from the proof of theorem 3.3 an algorithm, which out of the code of a proof of $\exists y P(y)$ computes an upper bound for the number n such that $\mathrm{T} \vdash P(\overline{n})$.*

Proof. This is clear because of corollary 4.1, and can be proven analogously to corollary 4.2. □

4.2 Establishing the numerical existence property for fragments of a recursively enumerable extension of arithmetic

Since the numerical existence property is implied by the disjunction property, it is interesting to investigate at which steps of the proof of theorem 3.3 the disjunction property was applied, and which further conclusions can be drawn from this.
In particular, it may be useful to know for which disjunctions of sentences the disjunction property of T is required to hold, to establish the numerical existence property for some given formula $P(y)$, or for a particular class of formulae (where as before T denotes a recursively enumerable extension of arithmetic, and $P(y)$ a formula with no free variable other than y). That is, whether T has to have the disjunction property for all disjunctions of sentences, or whether it is sufficient to have it only for some smaller class of sentences, and if so, then for which class. Of course this depends on the formula $P(y)$ or the class of formulae for which one wants to establish the numerical existence property.
Also, there are different ways to divide formulae into classes. Two different classifications are analyzed here.

4.2.1 Establishing the numerical existence property for prenex formulae with a limited number of quantifiers

Throughout this subsection let T, $P(y)$, $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$, $A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$ be defined as in the proof of theorem 3.3.
A well-known classification of formulae distinguishes formulae depending on the number of blocks of equal quantifiers.

Definition 4.4 (The formula-classes Σ^0_n,Π^0_n)**.** Let T be an extension of arithmetic. A formula A in the language of T is called Σ^0_n-formula, if it is of the form $\exists \underline{x_1} \forall \underline{x_2} \ldots \exists/\forall \underline{x_n} A_0\left(\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}\right)$, where $\underline{x_i}$ is a tuple of variables for each $i \in \{1, \ldots, n\}$, and $A_0\left(\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}\right)$ is a quantifier-free formula.
Likewise, if A has the form $\forall \underline{x_1} \exists \underline{x_2} \ldots \exists/\forall \underline{x_n} A_0\left(\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}\right)$, then A is called Π^0_n-formula.

The proof of theorem 3.3 establishes the numerical existence property for the formula $P(y)$. Now let us examine, for which class of formulae the disjunction property is required in the proof, if $P(y)$ is equivalent to a Σ^0_n- or a Π^0_n-formula.

Corollary 4.5 (Requirements to establish NEP for Σ^0_n- or Π^0_n-formulae)**.** *If* T *has the disjunction property for disjunctions of Σ^0_2-sentences, or alternatively for disjunctions of Π^0_2-sentences, then it has the numerical existence property for Σ^0_1-formulae.*
If $n \geq 2$ and T *obeys the disjunction property for disjunctions of Σ^0_n-sentences, then* T *has the numerical existence property for Σ^0_n-formulae, as well as for Π^0_{n-1}-formulae.*

Proof. Assume that $P(y)$ is equivalent to a Σ^0_n-formula or to a Π^0_n-formula. Then all sentences, which are parts of disjunctions on which the disjunction property is applied in the proof, are equivalent to some Σ^0_m- or some Π^0_m-formulae.
Let $P(y)$ be equivalent to a Σ^0_n-formula or to a Π^0_{n-1}-formula for some $n \geq 2$ (we have to consider the case that $n \leq 1$ separately). Then out of all sentences, which are parts of disjunctions on which the

disjunction property is applied in the proof, $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ (which depends on $P(y)$) is the sentence with the (strictly) greatest number of quantifiers. Thus it is sufficient for T to have the disjunction property for a class of sentences, that includes sentences with as many or fewer blocks of equal quantifiers as in $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$.
The formula $A_P(\overline{\mathrm{sub}}\left(\overline{k}\right))$ denotes

$$\exists y\left(\left(\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\vee P\left(y\right)\right)\wedge\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)\right).$$

We can get rid of the universal quantifier in $A_P(\overline{\mathrm{sub}}\left(\overline{k}\right))$ by the use of bounded minimalization.
Let $g(y):=\mu_{z\leq y}\left(\mathrm{prf}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\right)$. Then it holds

$$\mathrm{T}\vdash\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)\leftrightarrow\overline{\mathrm{prf}}_T\left(\overline{g}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}\left(y\right)=y. \tag{4.1}$$

It clearly holds $\mathrm{T}\vdash\forall z\left(\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)=0\rightarrow y\leq z\right)\leftrightarrow\forall z\left(z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right)$, and therefore the following two derivations yield (4.1).

$$\dfrac{\dfrac{\dfrac{\text{lemma 2.7 (i)}}{\overline{g}(y)\leq y}}{\overline{g}(y)<y\vee\overline{g}(y)=y}\qquad\dfrac{\dfrac{\dfrac{\forall z\left(z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right)}{\overline{g}(y)<y\rightarrow\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}\quad[\overline{g}(y)<y]}{\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}(y)=y}\qquad\dfrac{[\overline{g}(y)=y]}{\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}(y)=y}}{\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}(y)=y}\ (\vee\text{-E})$$

$$\dfrac{\dfrac{\dfrac{[\overline{g}(y)=y]^{(2)}}{z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}\ (\text{lemma 2.7 (iii)})}{\forall z\left(z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right)}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\left[\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right]^{(2)}}{\overline{g}(y)=0}\ (\text{lemma 2.7 (iv)})\quad\dfrac{[z<y]^{(1)}}{\overline{g}(z)\leq\overline{g}(y)}\ (\text{lemma 2.7 (v)})}{\overline{g}(z)=\overline{g}(y)}}{\overline{\mathrm{prf}}_T\left(\overline{g}(z),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}}{\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}\ (\text{lemma 2.7 (ii)})}{z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0}\ (\rightarrow\text{-I})^{(1)}}{\forall z\left(z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right)}}{\left(\overline{\mathrm{prf}}_T\left(\overline{g}(y),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}(y)=y\right)\rightarrow\forall z\left(z<y\rightarrow\overline{\mathrm{prf}}_T\left(z,\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\right)}\ (\vee\text{-E})^{(2)},\ (\rightarrow\text{-I})$$

Thus we have,

$$\begin{aligned}&\mathrm{T}\vdash A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\\&\quad\leftrightarrow\exists y\left(\left(\overline{\mathrm{prf}}_T\left(y,\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)=0\vee P\left(y\right)\right)\wedge\left(\overline{\mathrm{prf}}_T\left(\overline{g}\left(y\right),\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\neq 0\vee\overline{g}\left(y\right)=y\right)\right).\end{aligned}$$

Thus we get the following.

- If $P(y)$ is equivalent to a Σ^0_n-formula with $n\geq 2$, then $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ is equivalent to a Σ^0_n-sentence.
- If $P(y)$ is equivalent to a Π^0_{n-1}-formula with $n\geq 2$, then $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ is equivalent to a Σ^0_n-sentence.

This implies the assertion for formulae $P(y)$, which are equivalent to a Σ^0_n- or Π^0_{n-1}-formulae with $n\geq 2$. Nevertheless we still have to consider the case that $P(y)$ is equivalent to a Σ^0_n-formula for some $n\leq 1$. In this case, $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)$ does not have a number of quantifiers strictly greater than that of any other sentence, which is a part of some disjunction on which the disjunction property is applied in the proof. In this case the proof applies the disjunction property on quantifier-free sentences, Σ^0_1-sentences, and a Π^0_1-sentence. In particular, the disjunction property is applied on the sentence $A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\vee A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\vee$

$\neg A_Q\left(\overline{\text{sub}}\left(\bar{l}\right)\right)$. Since $A_Q\left(\overline{\text{sub}}\left(\bar{l}\right)\right)$ is equivalent to a Σ^0_1-sentences (which can be seen by the previous argumentation), the sentence $\neg A_Q\left(\overline{\text{sub}}\left(\bar{l}\right)\right)$ is equivalent to a Π^0_1-sentence. Thus T must have the disjunction property for a class of sentences, that contains disjunctions of Σ^0_1- and Π^0_1-sentences. That is, it is sufficient for T to have the disjunction property for disjunctions of Σ^0_2-sentences, or alternatively for disjunctions of Π^0_2-sentences. □

The following remarks discuss the optimality of the result acquired in corollary 4.5 for Σ^0_n-formulae with $n \in \{0, 1, 2\}$.

Remark 4.6 (Σ^0_0-DP $\not\Rightarrow$ Σ^0_0-NEP)**.** Opposed to what corollary 4.5 states, assume that any recursively enumerable extension of arithmetic T, which fulfills the disjunction property for disjunctions of quantifier-free sentences, also obeys the numerical existence property for quantifier-free formulae. Let PA* denote PA with the sentence $\exists x\left(\overline{\text{prf}}_{\text{PA}}\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right)$ as an additional axiom, where $\text{prf}_{\text{PA}}\left(x, y\right)$ denotes the provability function of PA, which "does not know" that $\exists x\left(\overline{\text{prf}}_{\text{PA}}\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right)$ is an axiom.
On the one hand, if PA* is consistent, then it obeys the disjunction property for disjunctions of quantifier-free sentences, because PA is Σ^0_1-complete (see 7.4.20 in [Dal04]) and obeys the law of excluded middle. Now, by assumption it follows that PA* fulfills the numerical existence property for quantifier-free formulae. Therefore and because of $\text{PA}^* \vdash \exists x\left(\overline{\text{prf}}_{\text{PA}}\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right)$, we have $\text{PA}^* \vdash \overline{\text{prf}}_{\text{PA}}\left(\bar{n}, \overline{\ulcorner\bot\urcorner}\right) = 0$ for some number n. But then $\text{PA} \vdash \overline{\text{prf}}_{\text{PA}}\left(\bar{n}, \overline{\ulcorner\bot\urcorner}\right) = 0$, because PA* is assumed to be consistent and PA has the disjunction property for quantifier-free sentences as well as the law-of-excluded-middle schema. Consequently it holds $\text{prf}_{\text{PA}}(n, \ulcorner\bot\urcorner) = 0$, and therefore $\text{PA} \vdash \bot$, that is, PA is inconsistent.
On the other hand, if PA* is inconsistent, then $\text{PA}^* \vdash 0 \neq 0$. Consequently it holds $\text{PA} \vdash \exists x\left(\overline{\text{prf}}_{\text{PA}}\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right) \to 0 \neq 0$. Now by the contrapositive of this statement and by $\text{PA} \vdash 0 = 0$ it follows $\text{PA} \vdash \neg\exists x\left(\overline{\text{prf}}_{\text{PA}}\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right)$. But then PA is inconsistent by Gödels second incompleteness theorem.
Thus the disjunction property for disjunctions of quantifier-free sentences in general does not imply the numerical existence property for quantifier-free formulae.

Remark 4.7 (Optimality of the result of corollary 4.5 for Σ^0_2-formulae)**.** Opposed to what corollary 4.5 states, assume that any recursively enumerable extension of arithmetic T, which fulfills the disjunction property for disjunctions of Σ^0_1-sentences, obeys the numerical existence property for Σ^0_2-formulae.
PA has the disjunction property for disjunctions of Σ^0_1-sentences, because PA is Σ^0_1-complete (see 7.4.20 in [Dal04]). Thus by assumption PA has the numerical existence property for Σ^0_2-formulae. Let $A(x)$ denote the Σ^0_2-formula

$$\forall y\left(\overline{\text{prf}}_T\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0 \vee \overline{\text{prf}}_T\left(y, \overline{\ulcorner\bot\urcorner}\right) \neq 0\right).$$

Then by the law-of-excluded-middle schema it holds $\text{PA} \vdash \exists x A(x)$, because

$$\text{PA} \vdash \exists x A(x) \leftrightarrow \exists x\left(\overline{\text{prf}}_T\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right) \vee \forall y\left(\overline{\text{prf}}_T\left(y, \overline{\ulcorner\bot\urcorner}\right) \neq 0\right),$$

that is,

$$\text{PA} \vdash \exists x A(x) \leftrightarrow \exists x\left(\overline{\text{prf}}_T\left(x, \overline{\ulcorner\bot\urcorner}\right) = 0\right) \vee \neg\exists y\left(\overline{\text{prf}}_T\left(y, \overline{\ulcorner\bot\urcorner}\right) = 0\right).$$

Since $\text{PA} \vdash \exists x A(x)$ holds, it follows by the numerical existence property for Σ^0_2-sentences that there is a number n such that $\text{PA} \vdash A(\bar{n})$. That is,

$$\text{PA} \vdash \overline{\text{prf}}_T\left(\bar{n}, \overline{\ulcorner\bot\urcorner}\right) = 0 \vee \forall y\left(\overline{\text{prf}}_T\left(y, \overline{\ulcorner\bot\urcorner}\right) \neq 0\right). \tag{4.2}$$

Assuming the consistency of PA we have $\text{PA} \vdash \overline{\text{prf}}_T\left(\bar{n}, \overline{\ulcorner\bot\urcorner}\right) \neq 0$ for all $n \in \mathbb{N}_0$. With (4.2) This implies

$$\text{PA} \vdash \forall y\left(\overline{\text{prf}}_T\left(y, \overline{\ulcorner\bot\urcorner}\right) \neq 0\right).$$

But this is a contradiction to Gödel's second incompleteness theorem.
Hence the requirement concerning the numerical existence property for Σ^0_2-formulae, which is given in corollary 4.5, is indeed optimal.

4.2.2 Heyting arithmetic as an interesting special case

This subsection gives an analysis of the relation between the disjunction and the numerical existence properties of HA. Interestingly HA behaves rather differently than the general case, insofar as assuming its own disjunction property for disjunctions of Σ_2^0-sentences it proves its own numerical existence property (for all formulae with a single free variable).

Corollary 4.8 (Due to [Lei85])**.** HA *proves the following: "If* HA *fulfills the disjunction property for disjunctions of* Σ_2^0*-sentences, then* HA *fulfills the numerical existence property".*

Proof. A proof is given later in this section. □

In [Lei85] Leivant outlines among Friedman's previously published theorems also some of Friedman's unpublished results. Essential for the proof of corollary 4.8, which is given in section 4 in [Lei85], is the concept of q-realizability, and the soundness theorem for q-realizability. Hence these are introduced first in the following. Then a proof of corollary 4.8 is given.

In order to treat q-realizability, elementary recursion theory must be formalized in HA. I will not give such a formalization in detail here. The next lemma presents without a proof some important results of such a formalization, which are needed in the current discussion. A suitable formalization of elementary recursion theory in HA is worked out in detail in section 7 of chapter 3 in [Tro88].

Lemma 4.9 (Kleene's T-predicate and the result extracting function)**.** *There exists a primitive recursive predicate* $\mathrm{T}\left(e, \underline{x}, n\right)$ *in the language of* HA*, which expresses that n codes a terminating computation sequence of the partial recursive function with code e and input* $\underline{x}$ *(*$\underline{x}$ *denotes a tuple). Also, there is a primitive recursive function* $\mathrm{U}(n)$*, which extracts the result of the computation sequence with code n. Usually,* T *is called* Kleene's T-predicate *and* U *the* result extracting function*. Furthermore, it holds that*

$$\mathrm{HA} \vdash T\left(x, \underline{y}, w\right) \wedge T\left(x, \underline{y}, z\right) \rightarrow w = z.$$

Proof. See section 7 of chapter 3 in [Tro88]. □

In the following only the function symbol of U will be needed (and not the function $\mathbb{N}_0 \rightarrow \mathbb{N}_0$ as such). So it will not lead to misunderstandings if the function symbol of U is denoted by U as well.
Furthermore, in the following j denotes a pairing function $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$, and j_1, j_2 denote functions $\mathbb{N}_0$ onto $\mathbb{N}_0$, such that

$$\mathrm{j}_1\left(\mathrm{j}(x,y)\right) = x, \quad \mathrm{j}_2\left(\mathrm{j}(x,y)\right) = y, \quad \mathrm{j}\left(\mathrm{j}_1(z), \mathrm{j}_2(z)\right) = z.$$

We can assume that the functions j, j_1 and j_2 are primitive recursive. For instance one can take

$$\begin{aligned}
\mathrm{j}(x,y) &:= \frac{1}{2}\left(x+y\right)\left(x+y+1\right)+x, \\
\mathrm{j}_1(z) &:= \mu_{x \le z}\left(\prod_{y \le z}\left(\mathrm{j}(x,y)-z\right)=0\right), \\
\mathrm{j}_2(z) &:= \mu_{y \le z}\left(\mathrm{j}\left(\mathrm{j}_1(z),y\right)-z=0\right),
\end{aligned}$$

which are primitive recursive functions.

Definition 4.10 (q-realizability predicate)**.** Let $A\left(x_0,\ldots,x_n\right)$ be a formula containing at most $x_0,\ldots,x_n$ free ($n \in \mathbb{N}_0$). By induction on the logical complexity of A the following defines the formula $x\mathrm{q}A\left(x_0,\ldots,x_n\right)$, where $x \notin \left\{x_0,\ldots,x_n\right\}$.

(i) $x\mathrm{q}A :\equiv A$ for A prime,

(ii) $x\mathrm{q}(A \wedge B) :\equiv \left(\bar{\mathrm{j}}_1(x)\mathrm{q}A\right) \wedge \left(\bar{\mathrm{j}}_2(x)\mathrm{q}B\right)$,

(iii) $x\mathrm{q}(A \vee B) :\equiv \left(\bar{\mathrm{j}}_1(x) = 0 \rightarrow \left(\bar{\mathrm{j}}_2(x)\mathrm{q}A\right) \wedge A\right) \wedge \left(\bar{\mathrm{j}}_1(x) \neq 0 \rightarrow \left(\bar{\mathrm{j}}_2(x)\mathrm{q}B\right) \wedge B\right)$,

(iv) $x\mathrm{q}(A \rightarrow B) :\equiv \forall u\left(\left(u\mathrm{q}A\right) \wedge A \rightarrow \exists v\left(\mathrm{T}(x,u,v) \wedge \mathrm{U}(v)\mathrm{q}B\right)\right)$,

(v) $x\mathrm{q}\left(\exists y B(y)\right) :\equiv \left(\bar{\mathrm{j}}_2(x)\mathrm{q}B\left(\bar{\mathrm{j}}_1(x)\right)\right) \wedge B\left(\bar{\mathrm{j}}_1(x)\right)$,

(vi) $x\mathrm{q}\left(\forall y B(y)\right) :\equiv \forall y \exists z\left(\mathrm{T}\left(x,y,z\right) \wedge \mathrm{U}(z)\mathrm{q}B(y)\right)$.

$x\mathrm{q}A$ is called q-*realizability predicate* of A.

Theorem 4.11 (Soundness)**.** *Let A be a sentence.*

(i) *If* HA $\vdash A$ *then there exists a number n such that* HA $\vdash \overline{n}\mathrm{q}A$.

(ii) *Under the assumption that* HA *obeys the numerical existence property for* Σ_1^0*-formulae, the statement (i) can be proven in* HA.

Proof. In the following Σ_1^0-NEP is used as an abbreviation for the numerical existence property for Σ_1^0-formulae.
A proof of (i) is given in 3.2.4, [Tro73]. At first glance it is not clear whether that proof fulfills the condition given in (ii), because it rather informally uses Kleene bracket notation $\{\cdot\}(\cdot)$ and Lambda terms $\lambda x.t$ (for which there are in general no function symbols in the language of HA). Nevertheless it can be formalized in HA+Σ_1^0-NEP, as required in (ii), but then the proof would become quite lengthy. Therefore the following outlines the general strategy of the proof, and then gives only a small part in detail, in order to indicate why Σ_1^0-NEP is required.
After selecting an axiom system of HA, the proof proceeds by induction on the length of deductions. Since the statement (i) is only required to hold for sentences, it should be first considered, how deductions of sentences look like, that is, how deductions of open formulae can be excluded from the induction. In a few minutes of reflection it becomes clear that it is enough to consider universal closures of axiom schemata and of formulae to which a rule is applied. That is, for each instance F of an axiom, the existence of a number n must be proven such that HA $\vdash \overline{n}\mathrm{q}F^*$, where F^* denotes the universal closure of F. Likewise, for each application of a rule with assumptions $F_0,\dots,F_m$ and conclusion F it has to be shown that if HA $\vdash F_i^*$ and HA $\vdash \overline{n_i}\mathrm{q}F_i^*$ for some number n_i for each $i \in \{0,\dots,m\}$, then there exists a number n such that HA $\vdash \overline{n}\mathrm{q}F^*$ (where $F^*, F_0^*, \dots, F_m^*$ denote universal closures of $F, F_0, \dots, F_m$ respectively). Besides, it is clear that the outlined induction step already includes the induction beginning.
As already mentioned this induction proof is given in 3.2.4, [Tro73], and it only has to be formalized in HA+Σ_1^0-NEP. To illustrate how this can be done and in particular how Σ_1^0-NEP is applied, the treatment of the modus ponens rule is given in the following. For simplicity only its application to closed formulae is covered.
Let A and B be sentences. Let HA $\vdash A$, HA $\vdash A \rightarrow B$, and let n and m be numbers such that HA $\vdash \overline{n}\mathrm{q}A$ and HA $\vdash \overline{m}\mathrm{q}(A \rightarrow B)$. By definition of q-realizability it follows

$$\mathrm{HA} \vdash \forall u\left(\left(u\mathrm{q}A\right) \wedge A \rightarrow \exists v\left(\mathrm{T}(\overline{m},u,v) \wedge \mathrm{U}(v)\mathrm{q}B\right)\right).$$

Since HA $\vdash \overline{n}\mathrm{q}A \wedge A$ it follows

$$\mathrm{HA} \vdash \exists v\left(\mathrm{T}(\overline{m},\overline{n},v) \wedge \mathrm{U}(v)\mathrm{q}B\right). \tag{4.3}$$

It follows

$$\frac{\dfrac{\mathrm{HA} \vdash \exists v \mathrm{T}(\overline{m},\overline{n},v)}{\mathrm{HA} \vdash \exists v\,(\mathrm{T}(\overline{m},\overline{n},v) \wedge \exists z\,(\mathrm{U}(v)=z))}}{\mathrm{HA} \vdash \exists z \exists v\,(\mathrm{T}(\overline{m},\overline{n},v) \wedge \mathrm{U}(v)=z)}$$

The predicate $\mathrm{T}(x,y,z)$ can be constructed as a quantifier-free formula, as it is for instance worked out in detail in chapter 8 in [Sho93]. Hence by Σ_1^0-NEP the above implies that there exists a number k such that

$$\mathrm{HA} \vdash \exists v \left(\mathrm{T}(\overline{m},\overline{n},v) \wedge \mathrm{U}(v)=\overline{k}\right).$$

With statement (4.3) it follows

$$\cfrac{\cfrac{\cfrac{\cfrac{\exists v\,(\mathrm{T}(\overline{m},\overline{n},v) \wedge \mathrm{U}(v)\,\mathrm{q}B) \qquad \exists w \left(\mathrm{T}(\overline{m},\overline{n},w) \wedge \mathrm{U}(w)=\overline{k}\right)}{\exists v \exists w \left(\mathrm{T}(\overline{m},\overline{n},v) \wedge \mathrm{T}(\overline{m},\overline{n},w) \wedge \mathrm{U}(w)=\overline{k} \wedge \mathrm{U}(v)\,\mathrm{q}B\right)}\ \text{(lemma 4.9)}}{\exists v \exists w \left(v=w \wedge \mathrm{U}(w)=\overline{k} \wedge \mathrm{U}(v)\,\mathrm{q}B\right)}}{\exists w \left(\mathrm{U}(w)=\overline{k} \wedge \mathrm{U}(w)\,\mathrm{q}B\right)}}{\overline{k}\,\mathrm{q}B}$$

Hence $\mathrm{HA} \vdash \overline{k}\,\mathrm{q}B$ holds for some number k. □

Now the prerequisites for the proof of corollary 4.8 are established. So the proof can be finally presented.

Proof of corollary 4.8. This proof is due to Leivant (4.2.3 in [Lei85]). I informally talk about provability in HA in the following.
Let Σ_2^0-DP abbreviate the disjunction property for disjunctions of Σ_2^0-sentences, and let Σ_1^0-NEP abbreviate the numerical existence property for Σ_1^0-formulae.
Let $P(y)$ be a formula with no free variable other than y. Assume $\mathrm{HA} \vdash \exists y P(y)$.
Suppose that HA obeys Σ_2^0-DP. Then HA fulfills Σ_1^0-NEP (see corollary 4.5).[1] Hence the given proof of theorem 4.11 (i) can be carried out. Consequently one can prove in HA+Σ_2^0-DP that $\mathrm{HA} \vdash \overline{n}\,\mathrm{q}\,(\exists y P(y))$ for some number n. By definition of q-realizability this is equivalent to

$$\mathrm{HA} \vdash \left(\bar{\mathrm{j}}_2(\overline{n})\,\mathrm{q}P\left(\bar{\mathrm{j}}_1(\overline{n})\right)\right) \wedge P\left(\bar{\mathrm{j}}_1(\overline{n})\right).$$

Further, there is a number k such that $\mathrm{j}_1(n)=k$. Since j_1 is a primitive recursive function, it follows $\mathrm{HA} \vdash \bar{\mathrm{j}}_1(\overline{n})=\overline{k}$. Thus $\mathrm{HA} \vdash P\left(\overline{k}\right)$.
Hence the numerical existence property is established. □

4.2.3 Considerations based on a classification of formulae which is more appropriate for intuitionistic logic

In the subsection 4.2.1 prenex formulae were classified into Σ_n^0- and Π_m^0-formulae ($m,n \in \mathbb{N}_0$), and the criteria to establish the numerical existence property for these classes of formulae were discussed. Although the necessary criteria were shown, the whole discussion may still seem nonsatisfying. While extensions of (intuitionistic) arithmetic are treated in this thesis, the discussed classification is not as appropriate for intuitionistic logic as it is for classical logic. It is well-known that a formula of an extension of arithmetic (for instance HA) is in general not (intuitionistically) equivalent to a formula in prenex normal form, that is a Σ_n^0- or a Π_m^0-formula. So the discussed classification is only appropriate for extensions of arithmetic containing the classical logic (for instance PA). Furthermore, in [Weh96] it is shown that HA_0 plus induction schema for prenex formulae is Π_2^0-conservative over Π_2^0-induction. On

[1] In [Lei85] it is claimed that Σ_1^0-DP is already sufficient to establish Σ_1^0-NEP. Nevertheless it is not clear how to prove this.

the other hand, PA is Π^0_2-conservative over HA. This further demonstrates why the classification into Σ^0_n- and Π^0_m-formulae cannot be optimal in the context of intuitionistic logic.
The following presents a classification which is more appropriate for intuitionistic logic. After that, analogously as it is done in the subsection 4.2.1, the requirements to establish the numerical existence property for the defined classes of formulae are discussed.
The classification presented below is due to [Bur00]. In a sense it is an analogy of the Π^0_n-classification, adapted for HA. Similar to Σ^0_n or Π^0_n, it is again a hierarchy which is based on the complexity of formulae, though this time the complexity is not measured just by the number of quantifiers.

Definition 4.12 (The formula-classes Φ_n).

(i) $\Phi_0 := \Sigma^0_0 = \Pi^0_0$.

(ii) $\Phi_1 := \Sigma^0_1$.

(iii) $\Phi_2 := \Pi^0_2$.

(iv) For $n \geq 3$ the class Φ_n is inductively defined by the following.
  (1) $\Phi_{n-1} \subseteq \Phi_n$.
  (2) If $A \in \Phi_{n-1}$ and $B \in \Phi_n$, then $A \to B \in \Phi_n$.
  (3) If $A \in \Phi_n$, then $\forall x A(x) \in \Phi_n$.
  (4) If $A, B \in \Phi_n$, then $A \vee B, A \wedge B \in \Phi_n$.
  (5) If $A \in \Phi_{n-2}$, then $\exists x A(x) \in \Phi_n$.

The Φ_n-classification of formulae is in a sense an analogy of the Π^0_n-classification, because for $n \geq 2$ the Φ_n-fragment of a theory is classically equivalent to its Π^0_n-fragment.
Additionally, the Φ_n-classification seems indeed to be quite convenient for dealing with HA. In contrast to the Σ^0_n- and Π^0_n-classification, each formula of HA is equivalent to some Φ_n-formula.
These results, which are due to [Bur00], are given in the following lemmata.

Lemma 4.13. *Every formula in the language of* HA *is (intuitionistically) equivalent to a Φ_n-formula for a suitable $n \in \mathbb{N}_0$.*

Proof. The assertion follows by an easy induction on formulae. □

Lemma 4.14. *Let $n \geq 2$.*

(i) *Every Φ_n-formula is classically equivalent to a Π^0_n-formula.*

(ii) *Every Π^0_n-formula is classically equivalent to a Φ_n-formula.*

Proof. (i) We apply induction on n.
If $n = 2$, then $\Phi_n = \Pi^0_n$ by definition.
Let $n = 3$. One only needs to check the defining statements of Φ_3.
  (1) Obviously $\Phi_2 \subseteq \Pi^0_3$.
  (2) Let $A \in \Phi_2$ and $B \in \Phi_3$, such that B is classically equivalent to a Π^0_3-formula, then

$$(A \to B) \leftrightarrow \left(\forall \underline{x} \exists \underline{y} A_0\left(\underline{x}, \underline{y}\right) \to \forall \underline{x'} \exists \underline{y'} \forall \underline{z'} B_0\left(\underline{x'}, \underline{y'}, \underline{z'}\right)\right)$$
$$\leftrightarrow \forall \underline{x'} \exists \underline{x}, \underline{y'} \forall \underline{y}, \underline{z'}\left(A_0\left(\underline{x}, \underline{y}\right) \to B_0\left(\underline{x'}, \underline{y'}, \underline{z'}\right)\right),$$

where A_0 and B_0 are suitable quantifier-free formulae and $\underline{x}, \underline{y}, \underline{x'}, \underline{y'}, \underline{z'}$ are tuples. Thus $A \to B$ is classically equivalent to a Π^0_3-formula.

(3) If $B \in \Phi_3$ is classically equivalent to a Π^0_3-formula, then so is $\forall x B(x)$.

(4) If $A, B \in \Phi_3$ are classically equivalent to Π^0_3-formulae, then so are $A \vee B, A \wedge B$.

(5) If $A \in \Phi_1 = \Sigma^0_1$, then $\exists x A(x) \in \Phi_1 \subseteq \Phi_2 \subseteq \Phi_3$.

Let $n \geq 4$. Again one needs to check the defining statements of Φ_n. The treatment of (1)-(4) is similar to the case of $n = 3$. Only (5) is different. If $A \in \Phi_{n-2}$, then by induction hypothesis A is classically equivalent to a Π^0_{n-2}-formula. Therefore $\exists x A(x)$ is classically equivalent to a Σ^0_{n-1}-formula, and consequently also equivalent to a Π^0_n-formula.

(ii) The following is shown by induction on $k \geq 1$.

- Every Π^0_{2k}-formula is classically equivalent to a Φ_{2k}-formula.
- Every Π^0_{2k+1}-formula is classically equivalent to a Φ_{2k+1}-formula.

Let $k = 1$. Then $\Pi^0_{2k} = \Pi^0_2 = \Phi_{2k}$ by definition. Now let $A \in \Pi^0_{2k+1} = \Pi^0_3$. Then by classical logic we have,

$$A \leftrightarrow \forall \underline{x} \exists \underline{y} \forall \underline{z} A_0 \left(\underline{x}, \underline{y}, \underline{z} \right) \leftrightarrow \underbrace{\forall \underline{x} \neg \underbrace{\forall \underline{y} \exists \underline{z} \neg A_0 \left(\underline{x}, \underline{y}, \underline{z} \right)}_{\Phi_2}}_{\Phi_3},$$

where A_0 denotes a suitable quantifier-free formula and $\underline{x}, \underline{y}, \underline{z}$ are tuples. Hence for $k = 1$ every Π^0_{2k+1}-formula is equivalent to a Φ_{2k+1}-formula.

For the induction step let $k \geq 2$. By the induction hypothesis every $B \in \Pi^0_{2k}$ is equivalent to a Φ_{2k}-formula, and by classical logic we have,

$$\underbrace{\forall \underline{x} \exists \underline{y} B \left(\underline{x}, \underline{y} \right)}_{\Pi^0_{2(k+1)}} \leftrightarrow \underbrace{\forall \underline{x} \neg \underbrace{\forall \underline{y} \neg \underbrace{B \left(\underline{x}, \underline{y} \right)}_{\Phi_{2k}}}_{\Phi_{2k+1}}}_{\Phi_{2(k+1)}}.$$

Likewise, by the induction hypothesis every $C \in \Pi^0_{2k+1}$ is equivalent to a Φ_{2k+1}-formula, so it follows

$$\underbrace{\forall \underline{x} \exists \underline{y} C \left(\underline{x}, \underline{y} \right)}_{\Pi^0_{2(k+1)+1}} \leftrightarrow \underbrace{\forall \underline{x} \neg \underbrace{\forall \underline{y} \neg \underbrace{C \left(\underline{x}, \underline{y} \right)}_{\Phi_{2k+1}}}_{\Phi_{2(k+1)}}}_{\Phi_{2(k+1)+1}}.$$

□

Now assume that we want to prove the numerical existence property for Φ_n-formulae. The following corollary (of the proof of theorem 3.3) gives sufficient conditions for the disjunction property.

Corollary 4.15 (Requirement to establish NEP for Φ_n-formulae). *Let* T *be a recursively enumerable extension of arithmetic, and let $n \in \mathbb{N}_0$.*

(i) *If $n < 2$ and* T *has the disjunction property for disjunctions of Φ_2-sentences, then* T *has the numerical existence property for Φ_n-formulae.*

(ii) *If $n \geq 2$ and* T *has the disjunction property for disjunctions of Φ_{n+2}-sentences, then it has the numerical existence property for Φ_n-formulae.*

Proof. We proceed analogously as in the proof of corollary 4.5.
Let $P(y)$ be a formula with no free variable other than y. Let $n \geq 2$ and assume that $P(y)$ is equivalent to a Φ_n-formula. Again, the most complex (that is, in Φ_m with the strictly largest m) formula on which the disjunction property is applied in the proof of theorem 3.3 is $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$. As already seen in the proof of corollary 4.5 we have,

$$\begin{aligned} \mathrm{T} \vdash\, & A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right) \\ & \longleftrightarrow \exists y \big(\underbrace{\underbrace{\left(\overline{\text{prf}}_T\left(y, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right) = 0 \vee P\left(y\right)\right)}_{\Phi_n} \wedge \underbrace{\left(\overline{\text{prf}}_T\left(\overline{g}\left(y\right), \overline{\text{sub}}\left(\overline{k}\right)\right) \neq 0 \vee \overline{g}\left(y\right) = y\right)}_{\Phi_0}}_{\Phi_n} \big), \end{aligned}$$

where $g(y) := \mu_{z \leq y}\left(\text{prf}_T\left(z, \overline{\text{sub}}\left(\overline{k}\right)\right) = 0\right)$.
Thus by definition of Φ_n it follows that $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ is equivalent to a Φ_{n+2}-formula.
On the other hand, if $n \leq 1$ and $P(y)$ is equivalent to a Φ_n-formula, then $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$, which is equivalent to a Φ_1-formula in this case, is not the most complex formula on which the disjunction property is applied in the proof of theorem 3.3. Since $\neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ is equivalent to a Φ_2-formula, the disjunction property for disjunctions of Φ_2-sentences is required. □

Observation 5.1. *Let $C(x)$ be a formula with no free variables other than x and let B be a formula that does not contain x free. Then* $\mathrm{HA} \vdash \neg B \rightarrow \exists x C(x)$ *implies that there exists a number n such that* $\mathrm{HA} \vdash \neg B \rightarrow C(\overline{n})$.

Proof. It is well known that HA is closed under the independence-of-premise-rule for negated premises (see [Tro88], p. 138), that is, for any formula $C(x)$ and any formula B, that does not contain x free, we have that $\mathrm{HA} \vdash \neg B \rightarrow \exists x C(x)$ implies $\mathrm{HA} \vdash \exists x (\neg B \rightarrow C(x))$. Since HA also obeys the numerical existence property (see [Tro88], p. 142), it follows that there exists a number n such that $\mathrm{HA} \vdash \neg B \rightarrow C(\overline{n})$. □

For a recursively enumerable extension of arithmetic T assume that $\mathrm{T} \vdash B \rightarrow \exists x C(x)$ holds. Observation 5.1 raises questions about (the required conditions for) the existence of a number n such that $\mathrm{T} \vdash B \rightarrow C(\overline{n})$ holds. Is the existence of such a number ensured, regardless of the choice of T, B and $C(x)$? In fact it is easy to see that in general a recursively enumerable extension of arithmetic can not have the discussed property without fulfilling some further conditions. For instance PA is a recursively enumerable extension of arithmetic, which clearly does not fulfill that property. But also HA does not fulfill it, as the following observation shows.

Observation 5.2. *Assume that for any formula $C(x)$ with no free variables other than x and any formula B, that does not contain x free, from* $\mathrm{HA} \vdash B \rightarrow \exists x C(x)$ *follows that there exists a number n such that* $\mathrm{HA} \vdash B \rightarrow C(\overline{n})$. *Then* PA *is inconsistent.*

Proof. Clearly it holds $\mathrm{HA} \vdash \exists x \left(\overline{\mathrm{prf}_{\mathrm{PA}}}\left(x, \overline{\ulcorner \bot \urcorner}\right) = 0\right) \rightarrow \exists x \left(\overline{\mathrm{prf}_{\mathrm{PA}}}\left(x, \overline{\ulcorner \bot \urcorner}\right) = 0\right)$ (where $\mathrm{prf}_{\mathrm{PA}}(x, y)$ is the provability function of PA). Now assume that there is a number n such that

$$\mathrm{HA} \vdash \exists x \left(\overline{\mathrm{prf}_{\mathrm{PA}}}\left(x, \overline{\ulcorner \bot \urcorner}\right) = 0\right) \rightarrow \overline{\mathrm{prf}_{\mathrm{PA}}}\left(\overline{n}, \overline{\ulcorner \bot \urcorner}\right) = 0. \tag{5.1}$$

If $\mathrm{prf}_{\mathrm{PA}}(n, \ulcorner \bot \urcorner) = 0$, then $\mathrm{PA} \vdash \bot$. Hence PA is inconsistent in this case.
Otherwise, if $\mathrm{prf}_{\mathrm{PA}}(n, \ulcorner \bot \urcorner) \neq 0$, then $\mathrm{PA} \vdash \neg \left(\overline{\mathrm{prf}_{\mathrm{PA}}}\left(\overline{n}, \overline{\ulcorner \bot \urcorner}\right) = 0\right)$, and therefore by the contrapositive of (5.1) it holds $\mathrm{PA} \vdash \neg \exists x \left(\overline{\mathrm{prf}_{\mathrm{PA}}}\left(x, \overline{\ulcorner \bot \urcorner}\right) = 0\right)$. But that implies by Gödels second incompleteness theorem that PA is inconsistent. □

It is well-known that HA obeys the disjunction and the numerical existence properties (see [Tro88], p. 142). Hence (assuming the consistency of PA and of T) observation 5.2 shows that even if the extension of arithmetic T is recursively enumerable and has the disjunction and the numerical existence properties, it is in general false that for any formula $C(y)$ with y free and any formula B, which does not contain y free, it holds that whenever $\mathrm{T} \vdash B \rightarrow \exists y C(y)$ there exists a number n such that $\mathrm{T} \vdash B \rightarrow C(\overline{n})$.
Now what are the required conditions for this property to hold? Some requirements that would suffice are quite easy to see. For example, the proof of observation 5.1 makes it clear that if a recursively enumerable extension of arithmetic T obeys the disjunction property, it is sufficient to have the independence-of-premise-rule for the premise B, i.e. the rule stating that $\mathrm{T} \vdash B \rightarrow \exists y C(y)$ implies $\mathrm{T} \vdash \exists y \left(B \rightarrow C(y)\right)$. Furthermore, if T obeys the disjunction property, it is easy to see that the requirement $\mathrm{T} \vdash B \vee \neg B$ is also sufficient.

Example 5.3. PA has both, the independence-of-premise-rule and the law-of-excluded-middle schema. Moreover PA has the disjunction property for disjunctions of Σ^0_1-sentences, because it is Σ^0_1-complete (see 7.4.20 in [Dal04]). Using corollary 4.5 by the above argumentation it follows that for any Σ^0_1-formula $C(y)$ with only y free and any quantifier-free sentence B not containing y free, it holds that if $\mathrm{PA} \vdash B \rightarrow \exists y C(y)$ then there exists a number n such that $\mathrm{PA} \vdash B \rightarrow C(\overline{n})$. By the proof of observation

5.2 it follows that this is not the case if B is not quantifier-free. And this is also not the case if $C(y)$ is not a Σ_1^0-formula, which can be shown using the Π_1^0-formula

$$\forall y\left(\overline{\text{prf}}_T\left(x,\overline{\ulcorner\bot\urcorner}\right)=0\vee\overline{\text{prf}}_T\left(y,\overline{\ulcorner\bot\urcorner}\right)\neq 0\right)$$

as $C(y)$ (very similar to the proof of remark 4.7).

Another sufficient criterion for the discussed property to hold is presented in theorem 5.5. The theorem and its proof run parallel to the statement and proof of theorem 3.3. Nevertheless, because of the premise, some steps of the proof must be done syntactically now. Therefore the following lemma is crucial in the proof.

Lemma 5.4. *Let n be a number and let A be a formula. Then* $\mathrm{T}\vdash\overline{\text{prf}}_T\left(\overline{n},\overline{\ulcorner A\urcorner}\right)=0\rightarrow A$.

Proof. If $\text{prf}_T(n,\ulcorner A\urcorner)=0$, then $\mathrm{T}\vdash A$, and so $\mathrm{T}\vdash\overline{\text{prf}}_T\left(\overline{n},\overline{\ulcorner A\urcorner}\right)=0\rightarrow A$.
If $\text{prf}_T(n,\ulcorner A\urcorner)\neq 0$, then $\mathrm{T}\vdash\neg\left(\overline{\text{prf}}_T\left(\overline{n},\overline{\ulcorner A\urcorner}\right)=0\right)$, that is $\mathrm{T}\vdash\overline{\text{prf}}_T\left(\overline{n},\overline{\ulcorner A\urcorner}\right)=0\rightarrow\bot$, and therefore $\mathrm{T}\vdash\overline{\text{prf}}_T\left(\overline{n},\overline{\ulcorner A\urcorner}\right)=0\rightarrow A$. □

For a formula B let us abbreviate the discussed properties as follows.

DP_B: For all sentences A_1, A_2 holds:
if $\mathrm{T}\vdash B\rightarrow A_1\vee A_2$, then $\mathrm{T}\vdash B\rightarrow A_1$ or $\mathrm{T}\vdash B\rightarrow A_2$.

NEP_B: For any formula $P(y)$ with no free variable other than y holds:
if $\mathrm{T}\vdash B\rightarrow\exists y P(y)$, then there exists a number n such that $\mathrm{T}\vdash B\rightarrow P(\overline{n})$.

Theorem 5.5 ($\mathrm{NEP}_B\Leftrightarrow\mathrm{DP}_B$). *Let* T *be a recursively enumerable extension of arithmetic and let B be a formula. Then* T *fulfills* NEP_B *if and only if* T *fulfills* DP_B.

Proof. Let T be a recursively enumerable extension of arithmetic which obeys DP_B. One can show that T also obeys NEP_B by adapting the proof of theorem 3.3 to the present situation. One only has to replace every self reference in every self referential sentence, which is used in the proof, with a reference to an implication, which has B as the premise and the self referential sentence as the conclusion. This approach is presented in detail in the following.
Let $P(y)$ be a formula with no free variable other than y. Let $A_P(x)$ denote the formula

$$\exists y\left(\left(\overline{\text{prf}}_T\left(y,\overline{\text{imp}}\left(\overline{\ulcorner B\urcorner},\overline{\text{neg}}(x)\right)\right)=0\vee P(y)\right)\wedge\forall z\left(\overline{\text{prf}}_T\left(z,\overline{\text{imp}}\left(\overline{\ulcorner B\urcorner},x\right)\right)=0\rightarrow y\leq z\right)\right).$$

Let $k:=\ulcorner A_P\left(\overline{\text{sub}}(x)\right)\urcorner$. Then $\ulcorner A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner=\text{sub}(k)$.
Let $Q(y)$ denote the formula $\overline{\text{prf}}_T\left(y,\overline{\text{imp}}\left(\overline{\ulcorner B\urcorner},\overline{\text{sub}}\left(\overline{k}\right)\right)\right)=0$. Let $A_Q(x)$ be the same formula as $A_P(x)$, except that Q is substituted for P. As before, choose a number l such that $\ulcorner A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner=\text{sub}(l)$.
Additionally observe,

$$\begin{aligned}
\ulcorner B\rightarrow A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner&=\text{imp}\left(\ulcorner B\urcorner,\ulcorner A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\urcorner\right)=\text{imp}(\ulcorner B\urcorner,\text{sub}(k)),\\
\ulcorner B\rightarrow A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner&=\text{imp}\left(\ulcorner B\urcorner,\ulcorner A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner\right)=\text{imp}(\ulcorner B\urcorner,\text{sub}(l)),\\
\ulcorner B\rightarrow\neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner&=\text{imp}\left(\ulcorner B\urcorner,\text{neg}\left(\ulcorner A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\urcorner\right)\right)=\text{imp}\left(\ulcorner B\urcorner,\text{neg}(\text{sub}(l))\right).
\end{aligned}$$

Let f, g, h be defined by

$$\begin{aligned}
f(y)&:=\mu_{x\leq y}\left(\text{prf}_T\left(x,\text{imp}(\ulcorner B\urcorner,\text{sub}(k))\right)=0\right),\\
g(y)&:=\mu_{x\leq y}\left(\text{prf}_T\left(x,\text{imp}(\ulcorner B\urcorner,\text{sub}(l))\right)=0\right),\\
h(y)&:=\mu_{x\leq y}\left(\text{prf}_T\left(x,\text{imp}(\ulcorner B\urcorner,\text{sub}(k))\right)\cdot\text{prf}_T\left(x,\text{imp}\left(\ulcorner B\urcorner,\text{neg}(\text{sub}(l))\right)\right)=0\right).
\end{aligned}$$

The use of the present definitions of the functions f, g, h as well as the substitution of $\overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{sub}}\left(\overline{k}\right)\right)$, $\overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{sub}}\left(\overline{l}\right)\right)$ and $\overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{l}\right)\right)\right)$ for $\overline{\text{sub}}\left(\overline{k}\right)$, $\overline{\text{sub}}\left(\overline{l}\right)$ and $\overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ respectively in the derivations 1, 2, 3, 4, 5 from the proof of theorem 3.3 yield the following

$$\mathrm{T} \vdash \exists y P\left(y\right) \to A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right) \vee A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right) \vee \neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$$

(except of the mentioned changes the deduction of this formula can be carried out identically as in the proof of theorem 3.3).
Thus from $\mathrm{T} \vdash B \to \exists y P\left(y\right)$ follows $\mathrm{T} \vdash B \to A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right) \vee A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right) \vee \neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$. With the property DP_B this implies that either $\mathrm{T} \vdash B \to A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ or $\mathrm{T} \vdash B \to A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ or $\mathrm{T} \vdash B \to \neg A_Q\left(\overline{\text{sub}}\left(\overline{l}\right)\right)$ holds. In the following each of this three cases will be considered separately. So assume $\mathrm{T} \vdash B \to \exists y P\left(y\right)$.

First case: If $\mathrm{T} \vdash B \to A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$, then $\mathrm{T} \vdash B \to P\left(\overline{n}\right)$ for some n.

Assume $\mathrm{T} \vdash B \to A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$. Then there exists a number m, such that

$$\mathrm{T} \vdash \overline{\text{prf}}_T\left(\overline{m}, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{sub}}\left(\overline{k}\right)\right)\right) = 0.$$

Fix m. We have,

$$\dfrac{\dfrac{\dfrac{\left(\overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0 \vee P\left(y\right)\right) \wedge \forall z\left(\overline{\text{prf}}_T\left(z, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{sub}}\left(\overline{k}\right)\right)\right) = 0 \to y \leq z\right)}{\overline{\text{prf}}_T\left(\overline{m}, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{sub}}\left(\overline{k}\right)\right)\right) = 0 \to y \leq \overline{m}}\,(\wedge\text{-E}), (\forall\text{-E})}{y \leq \overline{m}}\,(\to\text{-E})}{\exists y\left(y \leq \overline{m} \wedge \left(\overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0 \vee P\left(y\right)\right)\right)}\,(\wedge\text{-I}), (\exists\text{-I})$$

Hence by the (∃-E)-rule applied to $A_P\left(\overline{\text{sub}}\left(\overline{k}\right)\right)$ it holds

$$\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge \left(\overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0 \vee P\left(y\right)\right)\right).$$

It follows

$$\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge \overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0\right) \vee \exists y\left(y \leq \overline{m} \wedge P\left(y\right)\right).$$

Thus by DP_B holds that either

$$\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge P\left(y\right)\right)$$

or

$$\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge \overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0\right).$$

First subcase: Suppose $\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge P\left(y\right)\right)$. By lemma 3.2 we have

$$\mathrm{T} \vdash y \leq \overline{m} \wedge P\left(y\right) \to \bigvee_{i \leq m} P\left(\overline{i}\right).$$

It follows that $\mathrm{T} \vdash B \to \bigvee_{i \leq m} P\left(\overline{i}\right)$ holds, and consequently $\mathrm{T} \vdash B \to P\left(\overline{i}\right)$ holds by DP_B for some $i \leq m$.
Second subcase: Suppose $\mathrm{T} \vdash B \to \exists y\left(y \leq \overline{m} \wedge \overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0\right)$.
By lemma 3.2 we have,

$$\mathrm{T} \vdash y \leq \overline{m} \wedge \overline{\text{prf}}_T\left(y, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0 \to \bigvee_{i \leq m} \overline{\text{prf}}_T\left(\overline{i}, \overline{\text{imp}}\left(\ulcorner B \urcorner, \overline{\text{neg}}\left(\overline{\text{sub}}\left(\overline{k}\right)\right)\right)\right) = 0.$$

Hence by the (∃-E)-rule $\mathrm{T} \vdash B \rightarrow \bigvee_{i \leq m} \overline{\mathrm{prf}}_T\left(\bar{i}, \overline{\mathrm{imp}}\left(\ulcorner B \urcorner, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right)\right) = 0$ holds, and consequently $\mathrm{T} \vdash B \rightarrow \overline{\mathrm{prf}}_T\left(\bar{i}, \overline{\mathrm{imp}}\left(\ulcorner B \urcorner, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right)\right) = 0$ holds by DP_B for some $i \leq m$.
Furthermore, by lemma 5.4 it holds

$$\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(\bar{i}, \overline{\mathrm{imp}}\left(\ulcorner B \urcorner, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right)\right) = 0 \rightarrow \left(B \rightarrow \neg A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right),$$

Thus we have,

$$\mathrm{T} \vdash B \rightarrow \left(B \rightarrow \neg A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right),$$

and consequently,

$$\mathrm{T} \vdash B \rightarrow \neg A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right).$$

But then $\mathrm{T} \vdash B \rightarrow \neg A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right) \wedge A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)$, i.e. $\mathrm{T} \vdash B \rightarrow \bot$. So $\mathrm{T} \vdash B \rightarrow P\left(\bar{n}\right)$ for any number n.
Hence in both subcases, $\mathrm{T} \vdash B \rightarrow P\left(\bar{n}\right)$ holds for some number n.

Second case: If $\mathrm{T} \vdash B \rightarrow A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$, then $\mathrm{T} \vdash B \rightarrow P\left(\bar{n}\right)$ for some n.

The same argumentation as in the first case, except that Q is substituted for P and l for k, yields $\mathrm{T} \vdash B \rightarrow Q\left(\bar{n}\right)$ for some number n; that is, $\mathrm{T} \vdash B \rightarrow \overline{\mathrm{prf}}_T\left(\bar{n}, \overline{\mathrm{imp}}\left(\ulcorner B \urcorner, \overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right) = 0$. Since by lemma 5.4 $\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(\bar{n}, \overline{\mathrm{imp}}\left(\ulcorner B \urcorner, \overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right) = 0 \rightarrow \left(B \rightarrow A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right)$ holds, it follows $\mathrm{T} \vdash B \rightarrow \left(B \rightarrow A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)\right)$, and therefore $\mathrm{T} \vdash B \rightarrow A_P\left(\overline{\mathrm{sub}}\left(\bar{k}\right)\right)$. Now the claim follows by the first case.

Third case: If $\mathrm{T} \vdash B \rightarrow \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$, then $\mathrm{T} \vdash B \rightarrow \bot$ (and consequently $\mathrm{T} \vdash B \rightarrow P\left(\bar{n}\right)$ for any number n).

Assume $\mathrm{T} \vdash B \rightarrow \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$. Then $\mathrm{prf}_T\left(m, \mathrm{imp}\left(\ulcorner B \urcorner, \mathrm{neg}(\mathrm{sub}(l))\right)\right) = 0$ for some number m. Fix m.
Suppose $\mathrm{prf}_T\left(i, \mathrm{imp}(\ulcorner B \urcorner, \mathrm{sub}(l))\right) = 0$ for some $i < m$. Then $\mathrm{T} \vdash B \rightarrow A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$.
Hence $\mathrm{T} \vdash B \rightarrow A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right) \wedge \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$, and so $\mathrm{T} \vdash B \rightarrow \bot$ in this case.
Otherwise, suppose $\mathrm{prf}_T\left(i, \mathrm{imp}(\ulcorner B \urcorner, \mathrm{sub}(l))\right) \neq 0$ for every $i < m$. Then

$$\mathrm{T} \vdash \bigwedge_{i<m} \overline{\mathrm{prf}}_T\left(\bar{i}, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) \neq 0.$$

Furthermore, by lemma 3.2 we have

$$\mathrm{T} \vdash \bigwedge_{i<m} \overline{\mathrm{prf}}_T\left(\bar{i}, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) \neq 0 \rightarrow \left(x < \bar{m} \rightarrow \overline{\mathrm{prf}}_T\left(x, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) \neq 0\right).$$

It follows that $\mathrm{T} \vdash x < \bar{m} \rightarrow \overline{\mathrm{prf}}_T\left(x, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) \neq 0$ holds. The contraposition-rule and the (∀-I)-rule applied to the contrapositive yield

$$\mathrm{T} \vdash \forall x \left(\overline{\mathrm{prf}}_T\left(x, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) = 0 \rightarrow \bar{m} \leq x\right).$$

Furthermore it holds $\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(\bar{m}, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right)\right) = 0$ by assumption.
Hence we have,

$$\mathrm{T} \vdash \left(\overline{\mathrm{prf}}_T\left(\bar{m}, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right)\right) = 0 \vee Q\left(\bar{m}\right)\right) \wedge \forall x \left(\overline{\mathrm{prf}}_T\left(x, \overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\mathrm{sub}}\left(\bar{l}\right)\right)\right) = 0 \rightarrow \bar{m} \leq x\right).$$

And therefore by the (∃-I)-rule it follows that $\mathrm{T} \vdash A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)$. Thus it holds

$$\mathrm{T} \vdash B \rightarrow A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right) \wedge \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right),$$

and consequently $\mathrm{T} \vdash B \rightarrow \bot$.

Thus $\mathrm{T} \vdash B \rightarrow P\left(\bar{n}\right)$ for some n in either case. That is, if T obeys DP_B, then T also obeys NEP_B.
The converse can be easily shown by the use of lemma 2.3. □

6 Unprovability of the disjunction property

Throughout this section let T be a recursively enumerable extension of arithmetic. Interestingly, we obtain as a consequence of theorem 3.3 the following: if T obeys the disjunction property and proves its own disjunction property, then it also proves its own inconsistency. After the proof of theorem 3.3 is formalized in HA_0 this interesting assertion follows by means of Löb's theorem.
First I prove Löb's theorem.

Theorem 6.1 (Löb's theorem)**.** *Let A be a sentence such that* $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \to A$.
Then it holds $\mathrm{T} \vdash A$.

Proof. Let A be a sentence and assume $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right) \to A$.
Let B denote the formula

$$\mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\mathrm{sub}}\left(\overline{\ulcorner \mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\mathrm{sub}}(x), \overline{\ulcorner A \urcorner}\right)\right) \urcorner}\right), \overline{\ulcorner A \urcorner}\right)\right).$$

Then it holds

$$\begin{aligned} \mathrm{sub}\left(\ulcorner \mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\mathrm{sub}}(x), \overline{\ulcorner A \urcorner}\right)\right) \urcorner\right) &= \ulcorner \mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\mathrm{sub}}\left(\overline{\ulcorner \mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\mathrm{sub}}(x), \overline{\ulcorner A \urcorner}\right)\right) \urcorner}\right), \overline{\ulcorner A \urcorner}\right)\right) \urcorner \\ &= \ulcorner B \urcorner. \end{aligned}$$

Thus by the identity axioms of T we have $\mathrm{T} \vdash B \longleftrightarrow \mathrm{Pr}_T\left(\overline{\mathrm{imp}}\left(\overline{\ulcorner B \urcorner}, \overline{\ulcorner A \urcorner}\right)\right)$. That is,

$$\mathrm{T} \vdash B \longleftrightarrow \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right). \tag{6.1}$$

By (D1) and (D3) it follows $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right) \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right)$, and with (D2) this implies

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right).$$

It follows

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right),$$

because $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right) \to \left(\mathrm{Pr}_T\left(\overline{\ulcorner B \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right)\right)$ by (D3).
So by the assumption on A it follows

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right) \to A.$$

Now this and (6.1) imply

$$\mathrm{T} \vdash B \to A. \tag{6.2}$$

An application of (D1) yields $\mathrm{T} \vdash \mathrm{Pr}_T\left(\overline{\ulcorner B \to A \urcorner}\right)$, and so by (6.1) it follows $\mathrm{T} \vdash B$. Hence $\mathrm{T} \vdash A$ holds by (6.2). □

We will also need the following useful lemma.

Lemma 6.2. *Let A be a sentence, and let f be a 1-ary primitive recursive function. Then*

$$\mathrm{T} \vdash \exists x \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right).$$

In particular, $\mathrm{T} \vdash \exists x \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{x}, \overline{\ulcorner A \urcorner}\right) = 0 \urcorner}\right) \to \mathrm{Pr}_T\left(\overline{\ulcorner A \urcorner}\right)$.

Proof. Let A be a sentence. It holds

$$\mathrm{T} \vdash \Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right). \tag{6.3}$$

To show (6.3) I follow the the proof, which is given in [Smo82], section 4.1.6.
Since $\mathrm{T} \vdash A \to \left(\overline{\mathrm{prf}}_T\left(\overline{f}(x), \overline{\ulcorner A \urcorner}\right) = 0 \to A\right)$, it holds by (D1*) and (D3*) that

$$\mathrm{T} \vdash \Pr_T\left(\ulcorner A \urcorner\right) \to \Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right). \tag{6.4}$$

Furthermore, it clearly holds $\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(\overline{f}(x), \overline{\ulcorner A \urcorner}\right) \neq 0 \to \left(\overline{\mathrm{prf}}_T\left(\overline{f}(x), \overline{\ulcorner A \urcorner}\right) = 0 \to A\right)$. So, by (D1*) and (D3*) it follows

$$\mathrm{T} \vdash \Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) \neq 0 \urcorner\right) \to \Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right). \tag{6.5}$$

Now the following derivation establishes (6.3).

$$\dfrac{\dfrac{\dfrac{\left[\overline{\mathrm{prf}}_T\left(\overline{f}(x), \overline{\ulcorner A \urcorner}\right) = 0\right]}{\Pr_T\left(\ulcorner A \urcorner\right)}\,(\exists\text{-I})}{\Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right)}\,(6.4) \qquad \dfrac{\dfrac{\left[\overline{\mathrm{prf}}_T\left(\overline{f}(x), \overline{\ulcorner A \urcorner}\right) \neq 0\right]}{\Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) \neq 0 \urcorner\right)}\,\otimes}{\Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right)}\,(6.5)}{\Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \to A \urcorner\right)}\,(\vee\text{-E})$$

where the step $\otimes$ uses lemma 2.12 on the 1-ary primitive recursive function with input x, which is equal to 0 if and only if $\mathrm{prf}_T\left(f(x), \ulcorner A \urcorner\right) \neq 0$.
Thus (6.3) is shown. An application of (D3*) to this statement now yields

$$\mathrm{T} \vdash \Pr_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}(\dot{x}), \overline{\ulcorner A \urcorner}\right) = 0 \urcorner\right) \to \Pr_T\left(\ulcorner A \urcorner\right).$$

This implies the assertion. □

Now I will formalize theorem 3.3 in T (in particular, it can be done in HA_0).

Lemma 6.3. *If* $\mathrm{T} \vdash \Pr_T\left(\ulcorner A \vee B \urcorner\right) \to \Pr_T\left(\ulcorner A \urcorner\right) \vee \Pr_T\left(\ulcorner B \urcorner\right)$ *for any sentences* A *and* B*, then* $\mathrm{T} \vdash \Pr_T\left(\ulcorner \exists y P(y) \urcorner\right) \to \exists w \Pr_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge P(y)\right) \urcorner\right)$ *for any formula* $P(y)$ *with no free variable other than* y.

Proof. The following is an adaptation of the proof of theorem 3.3.
Let $P(y)$ be a formula with no free variable other than y. As in the proof of theorem 3.3, let $A_P(x)$ denote the formula

$$\exists y\left(\left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}(x)\right) = 0 \vee P(y)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T(z, x) = 0 \to y \leq z\right)\right).$$

Again let $Q(y)$ denote the formula $\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0$, and let $A_Q(x)$ be the same formula as $A_P(x)$, except that Q is substituted for P. Also as in the proof of theorem 3.3, choose numbers k and l such that $\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner = \mathrm{sub}(k)$ and $\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right) \urcorner = \mathrm{sub}(l)$.
Then we have,

$$\mathrm{T} \vdash \exists y P(y) \to A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \vee A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right) \vee \neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right),$$

as shown in the proof of theorem 3.3 (denoted there by claim 1).
By (D1), (D3) and the assumption it follows

$$\begin{aligned} T \vdash \; & \mathrm{Pr}_T\left(\overline{\ulcorner \exists y P(y) \urcorner}\right) \\ & \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) \vee \mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right) \urcorner}\right) \vee \mathrm{Pr}_T\left(\overline{\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right) \urcorner}\right). \end{aligned} \tag{6.6}$$

The disjunction from (6.6) can be treated in the same manner as the distinction in the proof of theorem 3.3. Namely, analogous to the three cases from the proof of theorem 3.3, three statements $(*)$, $(**)$ and $(***)$ are shown in the following.

First I prove:

$$(*) \quad T \vdash \mathrm{Pr}_T\left(\overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) \rightarrow \exists w \mathrm{Pr}_T\left(\overline{\ulcorner \exists y\left(y \leq \dot{w} \wedge P(y)\right) \urcorner}\right).$$

We have,

$$\begin{array}{c} \left[\left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \vee P(y)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \rightarrow y \leq z\right)\right]^{(2)} \qquad \left[\overline{\mathrm{prf}}_T\left(w, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right]^{(1)} \\ \hline y \leq w \wedge \left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \vee P(y)\right) \\ \hline \left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \left(y \leq w \wedge P(y)\right) \\ \hline \exists y\left(\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \left(y \leq w \wedge P(y)\right)\right) \\ \hline \exists y\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \exists y\left(y \leq w \wedge P(y)\right) \\ \hline \overline{\mathrm{prf}}_T\left(w, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \rightarrow \exists y\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \exists y\left(y \leq w \wedge P(y)\right) \quad (\rightarrow\text{-I})^{(1)} \\ \hline A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \rightarrow \left(\overline{\mathrm{prf}}_T\left(w, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \rightarrow \exists y\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \exists y\left(y \leq w \wedge P(y)\right)\right) \quad (\exists\text{-E})^{(2)}, (\rightarrow\text{-I}) \end{array}$$

Hence by (D1*) and (D3*) it follows

$$\begin{aligned} T \vdash \; & \mathrm{Pr}_T\left(\overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) \\ & \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \left(\overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \rightarrow \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \vee \exists y\left(y \leq \dot{w} \wedge P(y)\right)\right) \urcorner}\right), \end{aligned}$$

and consequently by (D3*) and the assumption it holds

$$\begin{aligned} T \vdash \; & \mathrm{Pr}_T\left(\overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) \rightarrow \left(\mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \urcorner}\right)\right. \\ & \left. \rightarrow \mathrm{Pr}_T\left(\overline{\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner}\right) \vee \mathrm{Pr}_T\left(\overline{\ulcorner \exists y\left(y \leq \dot{w} \wedge P(y)\right) \urcorner}\right)\right). \end{aligned} \tag{6.7}$$

Let us proceed with another derivation.

$$\begin{array}{c} \begin{array}{c} \mathrm{Pr}_T\left(\overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) \\ \hline \exists w\left(\overline{\mathrm{prf}}_T\left(w, \overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) = 0\right) \end{array} \qquad \begin{array}{c} \left[\overline{\mathrm{prf}}_T\left(w, \overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) = 0\right] \\ \hline \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) = 0 \urcorner}\right) \quad (\text{lemma } 2.12) \\ \hline \exists w \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) = 0 \urcorner}\right) \end{array} \\ \hline \exists w \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner}\right) = 0 \urcorner}\right) \quad (\exists\text{-E}) \\ \hline \exists w \mathrm{Pr}_T\left(\overline{\ulcorner \overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0 \urcorner}\right) \end{array}$$

That is,

$$T \vdash \mathrm{Pr}_T\left(\ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right) \rightarrow \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{w}, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0} \urcorner\right). \tag{6.8}$$

Now (6.7) and (6.8) imply

$$\begin{aligned} T \vdash\ & \mathrm{Pr}_T\left(\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner\right) \\ & \rightarrow \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right) \urcorner\right) \vee \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right), \end{aligned} \tag{6.9}$$

Let $f(y) := \mu_{x \leq y}\left(\mathrm{prf}_T\left(x, \mathrm{neg}\left(\mathrm{sub}\left(k\right)\right)\right) = 0\right)$. Then it holds that

$$\begin{aligned} T \vdash\ & \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \\ & \rightarrow \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}\left(\dot{w}\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \urcorner\right), \end{aligned} \tag{6.10}$$

as is shown by the following.

$$\dfrac{\exists y\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \qquad \dfrac{\dfrac{\dfrac{\left[y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right]}{y \leq w \wedge \overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0}\text{(lemma 2.7 (ii))}}{\overline{f}\left(y\right) \leq \overline{f}\left(w\right) \wedge \overline{\mathrm{prf}}_T\left(\overline{f}\left(y\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0}\text{(lemma 2.7 (v))}}{\overline{\mathrm{prf}}_T\left(\overline{f}\left(w\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0}\text{(lemma 2.7 (iv))}}{\overline{\mathrm{prf}}_T\left(\overline{f}\left(w\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0}\text{(∃-E)}$$

Thus $T \vdash \exists y\left(y \leq w \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \rightarrow \overline{\mathrm{prf}}_T\left(\overline{f}\left(w\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0$. So (6.10) follows by (D1*) and (D3*).
With (6.10) we have,

$$\dfrac{T \vdash \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}\left(\dot{w}\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \urcorner\right)}{T \vdash \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}\left(\dot{w}\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \urcorner\right)}\text{(∃-I)}$$

$$\dfrac{T \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{f}\left(\dot{w}\right), \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0 \urcorner\right)}{T \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right)}\text{(lemma 6.2)}$$

$$\dfrac{T \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\ulcorner \neg A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner\right)}{\dfrac{T \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \rightarrow \bot \urcorner\right)}{T \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \rightarrow \exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right) \urcorner\right)}\text{(cor. 2.14)}}$$

Now (D3*) yields

$$\begin{aligned} T \vdash\ & \exists w \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)\right) = 0\right) \urcorner\right) \\ & \rightarrow \left(\mathrm{Pr}_T\left(\ulcorner A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right) \urcorner\right) \rightarrow \mathrm{Pr}_T\left(\ulcorner \exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right) \urcorner\right)\right). \end{aligned} \tag{6.11}$$

Statements 6.9 and 6.11 finally imply $(*)$.

The next important statement is

$$(**) \quad \mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right)} \urcorner\right).$$

Let us make $(**)$ clear.
By the definition of $A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)$ and by the proof of $(*)$ we have,

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right). \tag{6.12}$$

Let $g(y) := \mu_{x \leq y}\left(\mathrm{prf}_T\left(x, \mathrm{sub}(k)\right) = 0\right)$. Then it holds that

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\dot{w}\right), \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0} \urcorner\right),$$

which can be proven in exactly the same manner as (6.10).
It follows

$$\cfrac{\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\dot{w}\right), \ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right) = 0} \urcorner\right)}{\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\dot{w}\right), \ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right) = 0} \urcorner\right)} \ (\exists\text{-I})$$

$$\cfrac{\mathrm{T} \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\overline{g}\left(\dot{w}\right), \ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right) = 0} \urcorner\right)}{\mathrm{T} \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right)} \ (\text{lemma } 6.2)$$

$$\cfrac{\mathrm{T} \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner \overline{A_P\left(\overline{\mathrm{sub}}\left(\overline{k}\right)\right)} \urcorner\right)}{\mathrm{T} \vdash \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{sub}}\left(\overline{k}\right)\right) = 0\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right)} \urcorner\right)} \ (*)$$

This and (6.12) yield $(**)$.

Finally I prove:

$$(***) \quad \mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) \to \exists w \mathrm{Pr}_T\left(\ulcorner \overline{\exists y\left(y \leq \dot{w} \wedge P\left(y\right)\right)} \urcorner\right).$$

It holds that

$$\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) \to \exists y \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)} \urcorner\right) \tag{6.13}$$

by the following derivation.

$$\cfrac{\mathrm{Pr}_T\left(\ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) \qquad \cfrac{\cfrac{\left[\overline{\mathrm{prf}}_T\left(y, \ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) = 0\right]}{\exists y \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{y}, \ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) = 0} \urcorner\right)} \ (\text{lemma } 2.12)}{\exists y \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{y}, \ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) = 0 \vee Q\left(\dot{y}\right)} \urcorner\right)} \ (\text{cor. } 2.14)}{\cfrac{\exists y \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{y}, \ulcorner \overline{\neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)} \urcorner\right) = 0 \vee Q\left(\dot{y}\right)} \urcorner\right)}{\exists y \mathrm{Pr}_T\left(\ulcorner \overline{\overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)} \urcorner\right)}} \ (\exists\text{-E})$$

Let $h(y) := \mu_{x\le y}\left(\mathrm{prf}_T\left(x, \mathrm{sub}\left(l\right)\right) = 0\right)$. We have,

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         \dfrac{\left[\overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0\right]}{\overline{h}\left(y\right) = 0}\text{(lemma 2.7(iv))}
         \qquad
         \dfrac{\left[z \le y\right]}{\overline{h}\left(z\right) \le \overline{h}\left(y\right)}\text{(lemma 2.7(v))}
        }{\overline{h}\left(z\right) = \overline{h}\left(y\right)}
        \qquad
        \overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0
       }{\overline{\mathrm{prf}}_T\left(\overline{h}\left(z\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0}
      }{\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0}\text{(lemma 2.7(ii))}
     }{z \le y \to \overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0}
    }{\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y < z}
   }{\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z}
  }{\forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z\right)}
 }{\overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \to \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z\right)}
}{}
$$

By (D1*) and (D3*) it follows

$$
\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\overline{h}\left(\dot{y}\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to \dot{y} \le z\right)\urcorner\right).
$$

An application of lemma 2.12 yields

$$
\mathrm{T} \vdash \overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \to \mathrm{Pr}_T\left(\ulcorner \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to \dot{y} \le z\right)\urcorner\right). \tag{6.14}
$$

Using this statement the following derivation is carried out.

$$
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      \left[\mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)\urcorner\right)\right]^{(2)}
      \qquad
      \dfrac{\left[\overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0\right]^{(1)}}{\mathrm{Pr}_T\left(\ulcorner \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to \dot{y} \le z\right)\urcorner\right)}\text{(6.14)}
     }{\mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)\urcorner\right) \wedge \mathrm{Pr}_T\left(\ulcorner \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to \dot{y} \le z\right)\urcorner\right)}
    }{\mathrm{Pr}_T\left(\ulcorner \left(\overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to \dot{y} \le z\right)\urcorner\right)}\text{(cor. 2.14)}
   }{\mathrm{Pr}_T\left(\ulcorner \exists y\left(\left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(y\right)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z\right)\right)\urcorner\right)}\otimes
  }{\mathrm{Pr}_T\left(\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right)}
 }{\overline{\mathrm{prf}}_T\left(\overline{h}\left(\dot{y}\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \to \mathrm{Pr}_T\left(\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right)}\text{(}\to\text{-I)}^{(1)}
}{\exists y \mathrm{Pr}_T\left(\ulcorner \overline{\mathrm{prf}}_T\left(\dot{y}, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(\dot{y}\right)\urcorner\right) \to \left(\overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \to \mathrm{Pr}_T\left(\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right)\right)}\text{(}\exists\text{-E)}^{(2)}\text{, (}\to\text{-I)}
$$

where the step $\otimes$ uses (D1*) and (D3*) applied to the formula

$$
\begin{aligned}
&\left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(y\right)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z\right) \to \\
&\qquad \exists y\left(\overline{\mathrm{prf}}_T\left(y, \overline{\mathrm{neg}}\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\right) = 0 \vee Q\left(y\right)\right) \wedge \forall z\left(\overline{\mathrm{prf}}_T\left(z, \overline{\mathrm{sub}}\left(\overline{l}\right)\right) = 0 \to y \le z\right).
\end{aligned}
$$

This derivation and (6.13) yield

$$
\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right) \to \left(\overline{\mathrm{prf}}_T\left(\overline{h}\left(y\right), \overline{\mathrm{sub}}\left(\overline{l}\right)\right) \neq 0 \to \mathrm{Pr}_T\left(\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right)\right). \tag{6.15}
$$

Having this, it is easy to see that

$$
\mathrm{T} \vdash \mathrm{Pr}_T\left(\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right) \to \mathrm{Pr}_T\left(\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\overline{l}\right)\right)\urcorner\right). \tag{6.16}
$$

The following derivation makes this clear.

$$
\dfrac{
\dfrac{\mathrm{Pr}_T\left(\overline{\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right) \qquad \left[\overline{\mathrm{prf}}_T\left(\bar{h}\left(y\right),\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\neq 0\right]^{(1)}}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}\,(6.15)
\qquad
\dfrac{\dfrac{\dfrac{\left[\overline{\mathrm{prf}}_T\left(\bar{h}\left(y\right),\overline{\mathrm{sub}}\left(\bar{l}\right)\right)=0\right]^{(1)}}{\overline{\mathrm{prf}}_T\left(\bar{h}\left(y\right),\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)=0}}{\exists z\overline{\mathrm{prf}}_T\left(z,\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)=0}}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}
}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}\,(\vee\text{-E})^{(1)}
$$

Now using (6.16), the statement $(***)$ can be easily derived:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Pr}_T\left(\overline{\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}\,(6.16) \qquad \mathrm{Pr}_T\left(\overline{\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)\wedge \mathrm{Pr}_T\left(\overline{\ulcorner \neg A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)}}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)\wedge \mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\rightarrow\bot\urcorner}\right)}}{\mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\urcorner}\right)\wedge \mathrm{Pr}_T\left(\overline{\ulcorner A_Q\left(\overline{\mathrm{sub}}\left(\bar{l}\right)\right)\rightarrow\exists y\left(y\leq\dot{w}\wedge P\left(y\right)\right)\urcorner}\right)}\,(\text{cor. }2.14)}{\mathrm{Pr}_T\left(\ulcorner\exists y\left(y\leq\dot{w}\wedge P\left(y\right)\right)\urcorner\right)}\,(\mathrm{D3}^*)}{\exists w\mathrm{Pr}_T\left(\ulcorner\exists y\left(y\leq\dot{w}\wedge P\left(y\right)\right)\urcorner\right)}
$$

Thus the statements $(*)$, $(**)$ and $(***)$ are proven.

The assertion $\mathrm{T}\vdash\mathrm{Pr}_T\left(\ulcorner\exists yP(y)\urcorner\right)\rightarrow\exists w\mathrm{Pr}_T\left(\ulcorner\exists y\left(y\leq\dot{w}\wedge P\left(y\right)\right)\urcorner\right)$ follows from (6.6), $(*)$, $(**)$ and $(***)$. □

Now all the preparation is done, so the main result of this section can be presented.

Theorem 6.4. *Let* T *be a recursively enumerable extension of arithmetic, such that for all sentences* A, B *it holds* $\mathrm{T}\vdash\mathrm{Pr}_T\left(\ulcorner A\vee B\urcorner\right)\rightarrow\mathrm{Pr}_T\left(\ulcorner A\urcorner\right)\vee\mathrm{Pr}_T\left(\ulcorner B\urcorner\right)$.
Then $\mathrm{T}\vdash\mathrm{Pr}_T\left(\ulcorner\bot\urcorner\right)$. *If additionally* T *obeys the disjunction property, then* $\mathrm{T}\vdash\bot$.

Proof. By lemma 6.3 it holds

$$
\mathrm{T}\vdash\mathrm{Pr}_T\left(\ulcorner\mathrm{Pr}_T\left(\ulcorner\bot\urcorner\right)\urcorner\right)\rightarrow\exists w\mathrm{Pr}_T\left(\ulcorner\exists y\left(y\leq\dot{w}\wedge\overline{\mathrm{prf}}_T\left(y,\overline{\ulcorner\bot\urcorner}\right)=0\right)\urcorner\right). \tag{6.17}
$$

Let $f(y):=\mu_{x\leq y}\left(\mathrm{prf}_T\left(y,\ulcorner\bot\urcorner\right)=0\right)$. We have,

$$
\dfrac{\exists y\left(y\leq w\wedge\overline{\mathrm{prf}}_T\left(y,\overline{\ulcorner\bot\urcorner}\right)=0\right) \qquad \dfrac{\dfrac{\dfrac{\left[y\leq w\wedge\overline{\mathrm{prf}}_T\left(y,\overline{\ulcorner\bot\urcorner}\right)=0\right]}{y\leq w\wedge\overline{\mathrm{prf}}_T\left(\bar{f}\left(y\right),\overline{\ulcorner\bot\urcorner}\right)=0}\,(\text{lemma 2.7 (ii)})}{\bar{f}\left(y\right)\leq\bar{f}\left(w\right)\wedge\overline{\mathrm{prf}}_T\left(\bar{f}\left(y\right),\overline{\ulcorner\bot\urcorner}\right)=0}\,(\text{lemma 2.7 (v)})}{\overline{\mathrm{prf}}_T\left(\bar{f}\left(w\right),\overline{\ulcorner\bot\urcorner}\right)=0}\,(\text{lemma 2.7})}{\overline{\mathrm{prf}}_T\left(\bar{f}\left(w\right),\overline{\ulcorner\bot\urcorner}\right)=0}\,(\exists\text{-E})
$$

Thus $T \vdash \exists y \left(y \leq w \wedge \overline{\mathrm{prf}}_T \left(y, \overline{\ulcorner \bot \urcorner} \right) = 0 \right) \rightarrow \overline{\mathrm{prf}}_T \left(\overline{f}(w), \overline{\ulcorner \bot \urcorner} \right) = 0$. It follows by (D1*) and (D3*) that

$$T \vdash \mathrm{Pr}_T \left(\overline{\ulcorner \exists y \left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T \left(y, \overline{\ulcorner \bot \urcorner} \right) = 0 \right) \urcorner} \right) \rightarrow \mathrm{Pr}_T \left(\overline{\ulcorner \overline{\mathrm{prf}}_T \left(\overline{f}(\dot{w}), \overline{\ulcorner \bot \urcorner} \right) = 0 \urcorner} \right),$$

consequently by the (∃-I)-rule

$$T \vdash \mathrm{Pr}_T \left(\overline{\ulcorner \exists y \left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T \left(y, \overline{\ulcorner \bot \urcorner} \right) = 0 \right) \urcorner} \right) \rightarrow \exists w \mathrm{Pr}_T \left(\overline{\ulcorner \overline{\mathrm{prf}}_T \left(\overline{f}(\dot{w}), \overline{\ulcorner \bot \urcorner} \right) = 0 \urcorner} \right),$$

and so

$$T \vdash \exists w \mathrm{Pr}_T \left(\overline{\ulcorner \exists y \left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T \left(y, \overline{\ulcorner \bot \urcorner} \right) = 0 \right) \urcorner} \right) \rightarrow \exists w \mathrm{Pr}_T \left(\overline{\ulcorner \overline{\mathrm{prf}}_T \left(\overline{f}(\dot{w}), \overline{\ulcorner \bot \urcorner} \right) = 0 \urcorner} \right).$$

Now an application of lemma 6.2 gives

$$T \vdash \exists w \mathrm{Pr}_T \left(\overline{\ulcorner \exists y \left(y \leq \dot{w} \wedge \overline{\mathrm{prf}}_T \left(y, \overline{\ulcorner \bot \urcorner} \right) = 0 \right) \urcorner} \right) \rightarrow \mathrm{Pr}_T \left(\overline{\ulcorner \bot \urcorner} \right). \tag{6.18}$$

The statements (6.17) and (6.18) imply

$$T \vdash \mathrm{Pr}_T \left(\overline{\ulcorner \mathrm{Pr}_T \left(\overline{\ulcorner \bot \urcorner} \right) \urcorner} \right) \rightarrow \mathrm{Pr}_T \left(\overline{\ulcorner \bot \urcorner} \right).$$

The assertion $T \vdash \mathrm{Pr}_T \left(\overline{\ulcorner \bot \urcorner} \right)$ follows by theorem 6.1.

If additionally T has the disjunction property, then by theorem 3.3 T also obeys the numerical existence property. Hence the statement $T \vdash \mathrm{Pr}_T \left(\overline{\ulcorner \bot \urcorner} \right)$ implies that there exists a number n, such that $T \vdash \overline{\mathrm{prf}}_T \left(\overline{n}, \overline{\ulcorner \bot \urcorner} \right) = 0$ holds. Consequently $\mathrm{prf}_T (n, \ulcorner \bot \urcorner) = 0$, that is, $T \vdash \bot$ holds. □

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